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Generalized method of fundamental solutions (GMFS) for boundary value problems

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ABSTRACT

In order to cope with the instability of the method of fundamental solutions (MFS), which caused by source offset, source location, or a fictitious boundary, a generalized method of fundamental solutions (GMFS) is proposed. The crucial part of the GMFS is using a generalized fundamental solution approximation (GFSA), which adopts a bilinear combination of fundamental solutions to approximate, rather than the linear combination of the MFS. Then the numerical solution of the GMFS is decided by a group of offsets corresponding to an intervention-point diffusion (IPD), instead of the MFS' offset of a single source. To demonstrate the effectiveness of the proposed approach, five numerical examples are given. The results have shown that the GMFS is more accurate, stable, and has a better convergence rate than the traditional MFS.

1. Introduction

In recent years the method of fundamental solutions (MFS), a boundary meshless method, has attracted great attention for solving homogeneous differential equations [1–9]. The MFS is quite simple, efficient, and easy for implementation, and it avoids the singular integrals which is necessary in certain boundary meshless methods, such as BNM [10], LBIE [11], HBNM [12], BCM [13], and BFM [14]. Furthermore, it could be highly accurate and rapidly convergent when an appropriate offset is selected [15].

However, despite the effectiveness and simplicity of the MFS, there are still some outstanding theoretical and numerical issues to be addressed [16–18]. One of the main issues yet to be resolved is the choice of the offset. In the MFS, a fictitious boundary outward offset to the real boundary with a distance parameter d is required in order to define the source points outside the domain. The offset d is sensitive and vital to the accuracy of the MFS. It is possible that we could set a reasonable range for the offset based on experience. However, it is not always effective, because a good offset for a certain problem could be bad for another problem. Despite the intensive research, this "offset dilemma" has been an outstanding research topic for the MFS [19,20].

In the past, various approaches have been proposed to alleviate this difficulty in the MFS such as the BKM [21–23], BCM [24,25] and BPM [26,27]. Instead of using the singular fundamental solutions as used in the MFS, these methods use non-singular kernels or general solutions. As such, the source points can be located on the real boundary, and the fictitious boundary is not needed. However, it is difficult to find the non-singular kernels or general solutions for some practical problems. Even though the non-singular kernels or general solutions can be found, the accuracy is normally not very impressive.

Another proposed method worthy to mention is the non-singular method of fundamental solutions (NMFS) [28,29]. For this method, a desingularization technique is used to regularize the singularity of the fundamental solution. The source points would then be located at the real boundary, making the fictitious boundary not necessary. Nevertheless, drawbacks include the necessity for the boundary nodes be distributed regularly, desingularization for arbitrary problems may not be available, and the tedious desingularization procedure compromises the simplicity of the method.

The singular boundary method (SBM) [30–36] uses an origin intensity factor (OIF) to substitute the singularity allowing the fictitious boundary not to be necessary. However, choosing the OIFs is not a trivial process, and the given problem must be solved twice.

Moreover, a boundary distributed source (BDS) method [37] should be mentioned also. For the BDS method, the source points do not necessarily need to be offset, but they should be distributed. The singular fundamental solution is integrated firstly over the distributed source covering the source points. If the distributed source is a simple shape, such as a circle, then the singular integrals could be evaluated analytically. However, the singular integrals are not always analytical, and the solution is inaccurate near the boundary regions. An improved BDS method [38,39] uses a boundary-integral technique to determine the singular

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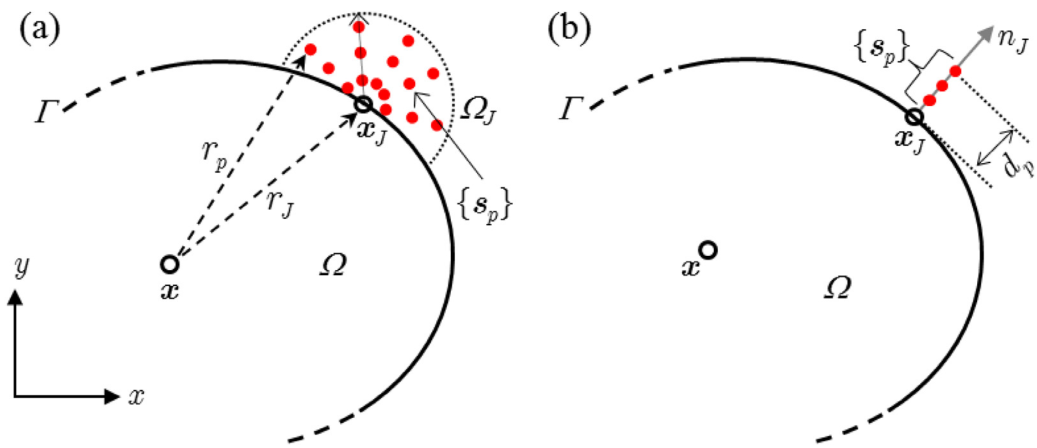

Fig. 1. Schematics of the GFSA: (a) regional IPD; (b) linear IPD.

integrals. However, this approach requires the singular integrals to be calculated directly.

In some sense the above-mentioned efforts overcome the old challenge of the MFS at a price of introducing new obstacles. From our understanding, the tenacious barrier of the MFS is still open for improvement. Thus, we will try to give another option for the issue of the MFS.

2. Generalized fundamental solution approximation

The MFS uses a fundamental solution approximation (FSA), which was first proposed by Kupradze and Aleksidze [1,40–42], as the basis function for solving homogeneous equations. It is notable that, another independent work with the same concept was also proposed by Wen [43] which is called the point intensity method (PIM). Let $u(\mathbf{x})$ be a field variable in a given domain Ω bounded by Γ. The basic idea of the FSA is to express $u(\mathbf{x})$ as a linear combination of fundamental solutions:

$$u(\mathbf{x}) = \sum_{J=1}^{N} a_J \psi_J(\mathbf{x},\mathbf{s}) \quad \mathbf{x} \in \bar{\Omega}, \tag{1}$$

where $\psi_J(\mathbf{x{,}s}) \equiv \psi(\mathbf{x},\ \mathbf{s}_J) \equiv \psi(r_J)$ is the fundamental solution, $r_J = \|\mathbf{x} - \mathbf{s}_J\|_2$ is the Euclidean norm between the measuring point $\mathbf{x}$ and the source point $\mathbf{s}_J$, a_J is the intensity coefficient at $\mathbf{s}_J$, and $\bar{\Omega} \equiv \Omega \cup \Gamma$.

Being different from the FSA, the generalized fundamental solution approximation (GFSA) uses a bilinear combination of fundamental solutions to approximate $u(\mathbf{x})$as follows:

$$u(\mathbf{x}) = \sum_{J=1}^{N} \sum_{p=1}^{N_p} a_J \psi_p^J(\mathbf{x},\mathbf{s}), \quad \{\mathbf{s}_p\} \in \Omega_J, \tag{2}$$

where $\{\mathbf{s}_p\} \notin \bar{\Omega}$ is the intervention-point diffusion (IPD) of the source node $\mathbf{x}_J$ [44], N_p is its point number, and Ω_J is the diffusion domain centered at $\mathbf{x}_J$ which is sheared off by the boundary Γ, as shown in Fig. 1(a). Note that we use a superscript "J" in the function ψ to denote a correspondence with the source node $\mathbf{x}_J$.

Note that the diffusion domain Ω_J could be arbitrarily selected outside the domain. For efficiency, we can also linearly diffuse the IPD, such as $\{\mathbf{s}_p\} \in n_J$ (which is not strict), as shown in Fig. 1(b) where n_J is the outward normal at $\mathbf{x}_J$, and d_p is the offset of an intervention point $\mathbf{s}_p$, as

$$d_p = \left\|\mathbf{s}_p - \mathbf{x}_J\right\|_2, \quad p = 1, 2, \ldots, N_p. \tag{3}$$

In this paper, a diffusion scheme to choose $\{\mathbf{s}_p\} \in n_J$ is our focus. The appropriate choice of $N_p(\geq 5$ is suggested) and offsets $\{d_p\}$ is necessary. By default, we choose

$$\{d_p\} = (0.1 : h_p : 0.5) \cdot \bar{R}, \tag{4}$$

where h_p is the interval for $\{d_p\}$ diffused, e.g., if $h_p = 0.1$, then $\{d_p\} = (0.1{,}0.2{,}0.3{,}0.4{,}0.5)\bar{R}$ for Eq. (4). $\bar{R}$ is defined as the normalized parameter for the boundary dimension

$$\bar{R} = \frac{1}{\sqrt{D}} \left\| \frac{\max(x_i) - \min(x_i)}{2} \right\|_2 \tag{5}$$

in which the subscript "i" is the component signal of the coordinates, and D is the number of dimensions.

Obviously, when $N_p = 1$, the GFSA is equivalent to the FSA. In other words, the GFSA is a generalized FSA. So we use a term of "generalized" to denote the novel approximation and the corresponding numerical method.

3. Generalized method of fundamental solutions (GMFS)

Consider the following Laplace equation in a 2D domain Ω bounded by Γ:

$$\nabla^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{6}$$

subject to boundary conditions (BCs):

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u, \tag{7}$$

$$u_{,n}(\mathbf{x}) \equiv \frac{\partial u}{\partial n}(\mathbf{x}) = \bar{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_t, \tag{8}$$

where Γ_u is the Dirichlet boundary, Γ_t is the Neumann boundary, $\Gamma = \Gamma_u \cup \Gamma_t$, $\Gamma_u \cap \Gamma_t = \emptyset$, n is the outward normal of the boundary, and $\bar{u}$, $\bar{q}$ are the known functions on the boundary. The fundamental solution ψ for the Laplacian is given by:

$$\psi(r) = \begin{cases} \dfrac{-1}{2\pi} \ln(r), & 2D, \\ \dfrac{1}{4\pi r}, & 3D. \end{cases} \tag{9}$$

The configuration of IPD (source-point cloud) of the GMFS is shown in Fig. 2(a). In contrast, the sources of the MFS are shown in Fig. 2(b). The target node $\mathbf{x}_I$ is indexed for constructing the discrete system equations. We will first try to use the variation method. The functional variation is given as

$$\delta\Pi_2 = \sum_{\mathbf{x}_I \in \Gamma_u} \delta u(u - \bar{u}) + \sum_{\mathbf{x}_I \in \Gamma_t} \delta u_{,n}\left(u_{,n} - \bar{q}\right). \tag{10}$$

Let $\delta\Pi_2 = 0$, then the BCs given by Eqs. (7) and (8) are satisfied. Then a GMFS1-type system of equations is obtained

$$\bar{\mathbf{K}}\mathbf{a} = \bar{\mathbf{F}}, \tag{11}$$