Engineering Analysis with Boundary Elements 94 (2018) 134–152

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Meshless analysis of parabolic interface problems

Masood Ahmad, Siraj-ul-Islam*

Department of Basic Sciences, Faculty of Architecture, Allied Sciences and Humanities, University of Engineering and Technology, Peshawar, Pakistan

ARTICLE INFO

Keywords:
Radial basis functions
Parabolic interface problem
Implicit scheme
Meshless collocation method
Scattered data points

ABSTRACT

In the current work, meshless methods are proposed to solve two-dimensional interface heat equation having closed interface boundary, in regular and irregular geometry as well as in regular and irregular interface settings. The current work extends applications of the conventional Kansa approach and the modified integrated RBF approach to numerical solution of interface PDE models. Accuracy of the meshless methods is confirmed through numerical experiments in both regular and irregular interface boundaries for a given set of problems. A set of scattered nodes (Halton points) is considered on both sides of the interface. Numerical evidence reveals accurate performance of the meshless methods for different test problems.

1. Introduction

Meshless methods remain the only choice among state of the art robust numerical methods for solving partial differential equations (PDEs), especially, in the situations where only scattered data is available in the complex shaped domain. Meshless methods can be accurately applied to variety of challenging PDEs embodying complex dynamics. Interface modeling gives birth to such complicated class of PDEs. Development of accurate numerical methods for such general type of interface problems arising from physical interface conditions containing jumps in a solution and its flux along the irregular boundary, is still at nascent stage.

Meshless methods are becoming increasingly popular for solving complex partial differential equations (PDEs), because of flexibility, ease of implementation and extension to higher dimensional complex geometry. Like other numerical methods, meshless methods establish numerical solution of PDEs, by forming a system of algebraic equations over the whole domain without using a pre-defined mesh or grid, for discretization. Meshless methods can use a set of either scattered or uniform nodes in the domain and on the boundaries. Other conventional methods such as finite-difference methods, finite-volume methods and finite-element methods require connected points, cells or elements in the form of mesh or grid to discretize the computational domain. These methods cannot be implemented on irregular geometries when only scattered data is available.

Various numerical meshless procedures have been developed in the literature for solving PDEs numerically. Among them are the meshless local Petrov–Galerkin method [1,2], the method of fundamental solution [3–6], the singular boundary method [7,8], the boundary knot method [9,10] and the radial basis collocation methods [11–15]. Meshless methods based on radial basis functions (RBFs) have gained much attention in recent years. In [16], a comparison of Multiqudric and its modified forms (integrated form) is discussed, which revealed that the integrated RBF may produce accurate results over a wide range of shape parameter. The RBFs that have been integrated several times appeared to be superior to the standard non-integrated RBFs, when the function being approximated is sufficiently smooth.

In integrated RBF method, the original RBF is integrated one or more times with respect to r to get new basis functions. When the original RBF is integrated n times and n is odd, the method is accurate and poorly conditioned for large values of shape parameter. But when n is even and the shape parameter is small, the method is accurate and poorly conditioned [17]. In the present study, we consider $n = 6$ i.e., the MQ RBF is integrated six times with respect to r. In different papers, integrated RBFs are applied to solve differential equations numerically. For instance, twice integrated MQ RBF with a fixed constant value of ϵ is used to solve one-dimensional boundary value problems [18]. A multi-domain integrated radial basis function collocation method is discussed for elliptic problems in [19]. In papers [20,21], the integrated RBF method is used to solve high-order ODEs and PDEs. More detail can be found in [16,22–24].

Numerical solutions of various interface parabolic and elliptic PDEs related to mathematical modeling of diffusion-transport related processes [25–32], are challenges ridden. Such type of PDEs have wide ranging applications in science and engineering. Parabolic interface problems of type (1) can be found in the food engineering, metal casting and hyperthermia therapy of tumor. Exact solution of such problems is often not available, especially when the interface is irregularly shaped. On the other hand, standard numerical methods often perform poorly for such time dependent interface problems, due to non-smooth or even discontinuous solution of the problem across the interfaces. Jump

* Corresponding author.
E-mail addresses: masood_suf@yahoo.com (M. Ahmad), siraj.islam@gmail.com (Siraj-ul-Islam).

https://doi.org/10.1016/j.enganabound.2018.06.008
Received 22 January 2018; Received in revised form 25 June 2018; Accepted 25 June 2018

conditions related to the solution and flux across the interface need to be incorporated in the numerical formulation in a proper way, to restore accuracy of the method in the interface region. If the jump conditions are not incorporated correctly in the numerical formulation, the standard numerical methods are destined to fail.

A meshless collocation method is reported in [29] for one-dimensional elliptic interface problems. A two-dimensional steady state heterogeneous conduction and Bioheat transfer problem with interface conditions [33] are investigated through radial basis collocation method. A steady-state linearized Poisson–Boltzmann problem, Poisson equation and steady-state Bioheat equation with interface conditions are solved numerically by meshless methods [12]. In [34], numerical solution of two-dimensional elastic wave equation is obtained through radial basis function. Numerical solution of heat transfer equilibrium problems having a smoothly curved interface are analyzed through radial basis function-generated finite-difference method [35].

In the present paper, meshless methods are used to solve two-dimensional parabolic interface problems with regular as well irregular outer and inner facial boundaries. Both integrated and conventional RBFs are used to collocate the solution on scattered and uniform nodal points. In some cases, we have used uniform nodal points for the purpose of comparison with the available numerical methods, whose performance is restricted to uniform nodal points only. For the current set of problems, accuracy of the integrated RBF collocation method is found better than the Kansa method due to less sensitivity to the shape parameter.

2. Governing equation

Consider a two-dimensional heat equation [31]

$$\frac{\partial}{\partial t}u(x,y,t) = \nabla.(\beta\nabla u(x,y,t)) + f(x,y,t), \quad \text{in} \quad \Omega = \Omega^+ \cup \Omega^-, \tag{1}$$

where $f(x, y, t)$ is a source term, $u(x, y, t)$ is the unknown function of interest, and β is the diffusion coefficient. The function $u(x, y, t)$ is given on the boundary $\partial\Omega$ as

$$u(x,y,t) = g(x,y,t), \quad \text{for} \quad (x,y) \in \partial\Omega. \tag{2}$$

The diffusion coefficient β in (1) is discontinuous across the material interface Γ separating two media Ω^+ and Ω^- i.e.,

$$\beta = \begin{cases} \beta^- & \text{in} \quad \Omega^-, \\ \beta^+ & \text{in} \quad \Omega^+. \end{cases}$$

The solutions $u^+(\mathbf{x})$ (in the solution domain Ω^+) and $u^-(\mathbf{x})$ (in the solution domain Ω^-) on both sides of the interface Γ are related through jump conditions as

$$\begin{aligned} [u] &= u^+(x,y,t) - u^-(x,y,t) = w(s,t), \quad \text{on} \quad \Gamma, \\ [\beta u_n] &= \beta^+\frac{\partial}{\partial n}u^+(x,y,t) - \beta^-\frac{\partial}{\partial n}u^-(x,y,t) = v(s,t), \quad \text{on} \quad \Gamma, \end{aligned} \tag{3}$$

where s is the arc-length parameterization of Γ and n is the unit normal direction.

3. The method

We approximate the time derivative in (1) by first-order forward-difference approximation

$$\frac{\partial}{\partial t}u(x,y,t) = \frac{u(x,y,t) - u(x,y,t_0)}{dt}, \tag{4}$$

where dt is the time-step, t_0 is the starting time of every time-step and $t = t_0 + dt$ be the next time value. We consider the following fully implicit formulation

$$u(x,y,t) - dt(\nabla.(\beta\nabla u(x,y,t))) = u(x,y,t_0) + dt\, f(x,y,t), \quad \text{in} \quad \Omega. \tag{5}$$

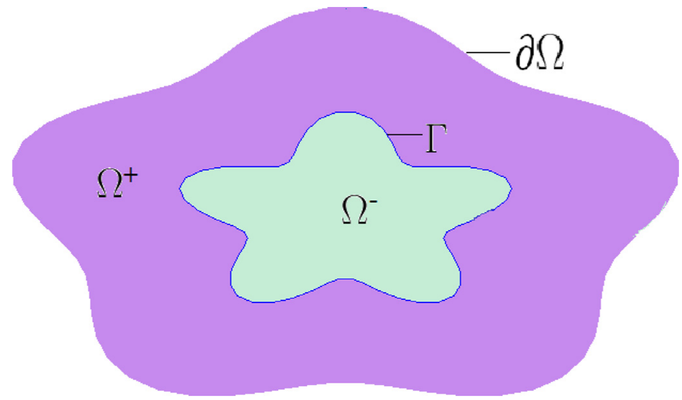

Fig. 1. A domain with irregular boundary and irregular interface.

3.1. Identification of collocation points

A set of scattered nodes is selected in the computational domain Ω and on the boundary $\partial\Omega$, which consist of four disjoint subsets (see Fig. 1). We assume that none of these subset is empty and $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$ be the sets of discrete collocation points in the sub-domains Ω^-, Ω^+, the boundary $\partial\Omega$ and the interface Γ, respectively. We further assume that the total number of collocation points in Ω^+, Ω^- , $\partial\Omega$ and Γ are N^+, N^-, K^+ and K^- respectively. Alternatively, they are represented as

$$\begin{aligned} \mathbf{x}_1 &= \left[\left(x_{1_1}, y_{1_1}\right), \left(x_{1_2}, y_{1_2}\right), \dots, \left(x_{1_{N^-}}, y_{1_{N^-}}\right)\right]^T \subset \Omega^-, \\ \mathbf{x}_2 &= \left[\left(x_{2_1}, y_{2_1}\right), \left(x_{2_2}, y_{2_2}\right), \dots, \left(x_{2_{N^+}}, y_{2_{N^+}}\right)\right]^T \subset \Omega^+, \\ \mathbf{x}_3 &= \left[\left(x_{3_1}, y_{3_1}\right), \left(x_{3_2}, y_{3_2}\right), \dots, \left(x_{3_{K^+}}, y_{3_{K^+}}\right)\right]^T \subset \partial\Omega, \\ \mathbf{x}_4 &= \left[\left(x_{4_1}, y_{4_1}\right), \left(x_{4_2}, y_{4_2}\right), \dots, \left(x_{4_{K^-}}, y_{4_{K^-}}\right)\right]^T \subset \Gamma, \end{aligned} \tag{6}$$

where the superscript T represents transpose of the vectors.

3.2. Radial basis functions

In the present work, two types of RBFs are chosen for numerical solution of parabolic interface problems [12]. These functions are defined as

$$\phi(\|r\|) = \begin{cases} \Big[\sqrt{1+(\epsilon\|r\|)^2}\left\{40(\epsilon\|r\|)^6 - 1518(\epsilon\|r\|)^4 + 1779(\epsilon\|r\|)^2 - 128\right\} + \\ \quad 105\epsilon\|r\| \sinh^{-1}(\epsilon\|r\|)\left\{8(\epsilon\|r\|)^4 - 20(\epsilon\|r\|)^2 + 5\right\}\Big]/201600\epsilon^6, \\ \sqrt{1+c^2\|r\|^2}, \end{cases} \tag{7}$$

where $\|r\|$ is a normed radial distance on a given dimension. The shape parameters c and ϵ control shapes of the respective radial basis functions. In RBF collocation method, the meshless solution of (1) is calculated separately in each sub-domain at time t as

$$\hat{u}(x,y,t) = \begin{cases} u^+(x,y,t) = \lambda_1(t)\phi_1^+(x,y) + \lambda_2(t)\phi_2^+(x,y) + \cdots \\ \quad +\lambda_{N_s^+}(t)\phi_{N_s^+}^+(x,y), \quad \text{in} \quad \Omega^+, \\ u^-(x,y,t) = \sigma_1(t)\phi_1^-(x,y) + \sigma_2(t)\phi_2^-(x,y) + \cdots \\ \quad +\sigma_{N_s^-}(t)\phi_{N_s^-}^-(x,y), \quad \text{in} \quad \Omega^-, \end{cases} \tag{8}$$

where N_s^- and N_s^+ are the number of source points in the sub-domains Ω^- and Ω^+, respectively. The collocation points and the source points are same in the present study. The function

$$\phi(x,y) = \begin{cases} \phi_i^+(x,y) = \phi^+(\|(x,y)-(x_i,y_i)\|_2), \quad i=1,2,3,\dots,N_s^+ \quad \text{in} \quad \Omega^+, \\ \phi_j^-(x,y) = \phi^-(\|(x,y)-(x_j,y_j)\|_2), \quad j=1,2,3,\dots,N_s^- \quad \text{in} \quad \Omega^-. \end{cases}$$