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A high accurate simulation of thin plate problems by using the method of approximate particular solutions with high order polynomial basis

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ABSTRACT

In this paper, a closed-form particular solution of polyharmonic splines has been obtained for high order partial differential operators. Instead of using complex derivation, the new particular solution is derived simply by adding or subtracting several available particular solutions. The proposed particular solution is further coupled with polynomial basis for numerically solving thin plate problems. The relationship between number of nodes and order of polynomials are fully studied. Numerical examples with irregular domains are presented to demonstrate the effectiveness of the proposed algorithm.

1. Introduction

During the last decade, radial basis functions (RBFs) have been successfully applied for solving various kinds of partial differential equations (PDEs) [3]. In contrast to the traditional meshed methods [1,2,19], the main attraction of the RBFs collocation methods is due to the simplicity of the solution procedure in which no tedious boundary and/or domain meshing is required. One category of the RBF based methods is the collocation method, which is simple and flexible with respect to the geometry of the domain [12]. However, the instability of collocation methods in dealing with derivatives has limited its ability in real applications. Many numerical techniques are proposed to enhance the instability of the RBF collocation method [9,22–25].

The RBF collocation process of solving PDEs can be done in two different ways. In the direct RBF collocation method, an RBF expansion is introduced with unknown coefficients for the solution of the PDE, then differentiated and collocated. On the other hand, the indirect RBF collocation can be done by integration instead of differentiation. The experimental data (inhomogeneous term) are interpolated by RBFs, then integrated in the polar coordinate system and collocated strictly on the governing equation and boundary conditions. The method of approximated particular solutions (MAPS) is a recently developed indirect RBF collocation method [4,5,21], which is slightly more accurate compared to the direct RBF collocation method [18]. These indirect methods need a closed-form particular solutions of the related RBFs.

Particular solutions are very popular in solving inhomogeneous equations for boundary type meshless methods, which split the partial differential equation (PDE) into homogeneous and inhomogeneous solutions [8,14–17,20]. The closed-form particular solutions for many commonly used RBFs of differential operators have been derived [7]. The derivation of particular solutions to Helmholtz-type equations using thin plate splines is firstly proposed by Chen and Rashed [6]. Then the concept was further extended to the polyharmonic splines [10,17]. Works based on particular solutions are mainly focus on simplifying the derivation of particular solutions for different PDEs; however, the derivation techniques of particular solutions sometimes are still too tedious to be applied for complex differential operators [20].

Instead of using the complex derivation perviously, a new closed-form particular solution of thin plate splines for high order harmonic differential operators are simply expressed by using the particular solutions of Laplacian and Helmholtz-type operators. The proposed particular solution is further coupled with polynomial for numerically solving high order PDEs based on the MAPS. The details of the nodes number and the order of polynomial are fully studied.

The organization of this paper is as follows. In Section 2, we introduce the formulation of the MAPS. The derivation of the new close particular solution are presented in Section 3. In Section 4, numerical examples of different problems are considered. Some conclusions of this paper with opening issues and future applications are given in the last section.

2. The formulation of the MAPS

Let us consider an thin plate vibration problem,

$$\left(\Delta^2 - \lambda^2\right)u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{1}$$

$$\Delta u(\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega, \tag{2}$$

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$$u(\boldsymbol{x}) = h(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial\Omega, \tag{3}$$

where λ is a non-zero constant, Δ is the Laplacian differential operator, f, g and h are given functions, Ω is bounded and closed domain with boundary $\partial\Omega$.

We know that the polynomial basis functions of degree less or equal than s in 2D case can be explicitly written as follows:

$$\begin{aligned}\{p_l\}_{l=1}^{q} &= \{x^{i-j}y^j : 0 \le j \le i, 0 \le i \le s\} \\ &= \{1, x, y, x^2, xy, y^2, \dots, x^s, x^{s-1}y, x^{s-2}y^2, \dots, xy^{s-1}, y^s\},\end{aligned} \tag{4}$$

where $q = (s+1)(s+2)/2$ is the number of polynomial basis functions with order s. Let $\{\boldsymbol{x}_j\}_{j=1}^{N_i}$ be a set of interior points in Ω and $\{\boldsymbol{x}_j\}_{j=N_i+1}^{N_i+N_b}$ be the boundary points on $\partial\Omega$ and $N = N_i + N_b$ be the total number of collocation points. In the MAPS, we assume the solution (1)–(3) can be approximated by the particular solutions of polyharmonic splines and polynomials basis in the following way:

$$u(\boldsymbol{x}) = \sum_{j=1}^{N} a_j \Phi(r_j) + \sum_{l=1}^{q} \beta_l p_l(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{5}$$

where a_j and β_l are the unknown coefficients, $r_j = \|\boldsymbol{x} - \boldsymbol{x}_j\|$ are the Euclidean norm, Φ is the RBF particular solution related to ϕ, which can be expressed as

$$(\Delta^2 - \lambda^2)\Phi(r) = \phi(r) = r^{2n}\ln(r), \tag{6}$$

where $\phi(r) = r^{2n}\ln(r)$ is the polyharmonic splines, and n denotes the order of polyharmonic splines. p_l is the particular solution of the augmented polynomial basis, which can be expressed as

$$(\Delta^2 - \lambda^2)p_l = x^{a-b}y^b, \qquad 0 \le b \le a, \quad 0 \le a \le s, \tag{7}$$

where s is the order of polynomial basis, x and y are coordinates of the point $\boldsymbol{x} = (x, y)$. By substituting (5) into (1), we have

$$Lu(\boldsymbol{x}_i) = \sum_{j=1}^{N} a_j \phi(r_{ij}) + \sum_{l=1}^{q} \beta_l Lp_l(\boldsymbol{x}_i), \qquad i = 1 \dots N_i, \tag{8}$$

where L is the differential operator $\Delta^2 - \lambda^2$. The standard polynomial insolvency constraint must be applied to deal with additional degrees of freedoms in (8)

$$\sum_{j=1}^{N} a_l Lp_l(\boldsymbol{x}_j) = 0, \qquad l = 1, 2, \dots q. \tag{9}$$

On the boundary we have

$$Bu(\boldsymbol{x}_i) = \sum_{j=1}^{N} a_j B\Phi(r_{ij}) + \sum_{l=1}^{q} \beta_l Bp_l(\boldsymbol{x}_i), \quad i = N_i + 1, \dots, N, \tag{10}$$

$$\sum_{j=1}^{N_b} a_{n_i+l} Bp_l(\boldsymbol{x}_j) = 0, \qquad l = 1, 2 \dots q, \tag{11}$$

where N_b is the number of the boundary nodes. Eqs (8)–(11) can be re-casted in a matrix form as

$$\begin{bmatrix} \boldsymbol{\phi}_{N_i \times N} & L\boldsymbol{p}_{N_i \times q} \\ \Delta\boldsymbol{\Phi}_{N_b \times N} & \Delta\boldsymbol{p}_{N_b \times q} \\ \boldsymbol{\Phi}_{N_b \times N} & \boldsymbol{p}_{N_b \times q} \\ [L\boldsymbol{p}^T_{N_i \times q}, \Delta\boldsymbol{p}^T_{N_b \times q}, \boldsymbol{p}^T_{N_b \times q}] & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{N \times 1} \\ \boldsymbol{\beta}_{q \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \\ \mathbf{0} \end{bmatrix}, \tag{12}$$

where the undetermined coefficients $\boldsymbol{a}$ and $\boldsymbol{\beta}$ can be obtained by solving the above linear system as long as the particular Φ and p_l are available. The formulated matrix is a non-square matrix since there are two types of boundary conditions in Eq. (12). In order to reduce the formulated matrix to square, two types of boundary nodes are used according to boundary conditions (2) and (3), as shown in Fig. 1.

The boundary condition (2) and (3) are, respectively, treated by considering red ● and blue ● boundary nodes as shown in Fig. 1. The formulated equation in (12) can be re-casted as

$$\begin{bmatrix} \boldsymbol{\phi}_{N_i \times N} & L\boldsymbol{p}_{N_i \times q} \\ \Delta\boldsymbol{\Phi}_{N_{b_1} \times N} & \Delta\boldsymbol{p}_{N_{b_1} \times q} \\ \boldsymbol{\Phi}_{N_{b_2} \times N} & \boldsymbol{p}_{N_{b_2} \times q} \\ [L\boldsymbol{p}^T_{N_i \times q}, \Delta\boldsymbol{p}^T_{N_{b_1} \times q}, \boldsymbol{p}^T_{N_{b_2} \times q}] & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_{N \times 1} \\ \boldsymbol{\beta}_{q \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \\ \mathbf{0} \end{bmatrix}, \tag{13}$$

where $N_b = N_{b_1} + N_{b_2}$, N_{b_1} and N_{b_2} denote the number of red ● and blue ● boundary nodes, respectively. The formulated matrix in (13) is transformed to a square matrix, and can be solved easily as long as the particular solutions are available.

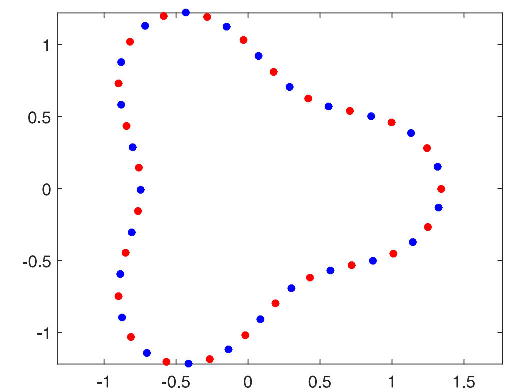

Fig. 1. Two different boundary nodes.

3. Particular solution of fourth order PDEs

The key of the success of the MAPS is the availability of the particular solution. The derivation of the particular solution is nontrivial and sometimes not even available, especially for higher order partial differential equations [20]. In this section, we introduce another way to evaluate the particular solution of the fourth order partial differential equations through the use of the particular solutions of the second order partial differential equations. First, we consider the particular solution of the following fourth order PDEs:

$$(\Delta^2 - \lambda^2)\Phi = \phi. \tag{14}$$

Instead of deriving the particular solution Φ from the above fourth order PDEs which could be tedious [20], we start with the particular solutions of Helmholtz and modified Helmholtz equations which are available in the Appendix.

$$(\Delta - \lambda)\Phi^m = \phi, \tag{15}$$

$$(\Delta + \lambda)\Phi^h = \phi. \tag{16}$$

After multiplying (16) by $\Delta - \lambda$ and (15) by $\Delta + \lambda$, subtracting (15) from (16) leads to

$$(\Delta + \lambda)(\Delta - \lambda)\Phi^m - (\Delta + \lambda)(\Delta - \lambda)\Phi^h = 2\lambda\phi, \tag{17}$$

it follows

$$(\Delta^2 - \lambda^2)\left(\frac{\Phi^m - \Phi^h}{2\lambda}\right) = \phi, \tag{18}$$

comparing (18) with (14), we can get

$$\Phi = \left(\frac{\Phi^m - \Phi^h}{2\lambda}\right). \tag{19}$$

The particular solution Φ in (14) can be expressed as a linear combination of particular solutions of modified Helmholtz Φ^m and Helmholtz Φ^h. Putting (19) to (2) leads to

$$\Delta\Phi = \frac{1}{2\lambda}(\Delta\Phi^m - \Delta\Phi^h), \tag{20}$$