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Modelling the third kind boundary condition in scaled boundary finite element method based numerical analysis

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ABSTRACT

Numerical models to deal with the third kind boundary condition (TKBC) are presented under the framework of Scaled Boundary Finite Element Method (SBFEM) for the 2-D static analysis. A gradient based algorithm is presented to tackle with nonlinear TKBC, and a temporally-piecewise adaptive algorithm is developed for a kind of time dependent TKBC. In addition, a proof that the appended stiffness matrix is block-circulant is given if the TKBC is cyclically symmetric, resulting in a reduction of computational expense of SBFEM based numerical analysis. Numerical examples are provided to verify proposed approaches, and satisfactory results are obtained.

1. Introduction

The third kind boundary condition usually describes relationships between variables and their derivatives along the boundary, and often appears in the simplification of modelling interactions between two structures, the impact of one structures on the another is often simplified as TKBC via linear/nonlinear or viscoelastic relationships between traction and displacement along the interface [1–6] Basically, numerical models to tackle with these boundary conditions are based on conventional finite element methods (FEM) using spring or spring-dashpot supports models, the contribution induced by TKBC can be described via an appended stiffness matrix [1,2]. When TKBC is nonlinear, simple iterative scheme based algorithms are usually employed to solve the nonlinear system equation, resulting in lower computing efficiency [3,4]. On the other hand, there seems almost no direct report concerned with the time dependent TKBC in the static analysis [5,6].

In the context of Scaled Boundary Finite Element Method (SBFEM) [7,8] that is semi-analytical, and exhibits good performance in dealing with problems of stress singularities and unbounded domains [9–14], but few works have been reported which attempts to deal with spring supports.

In this paper, regarding to different TKBC, different numerical models are presented under the framework of SBFEM. A gradient based algorithm is presented to tackle with nonlinear TKBC, and a temporally-piecewise adaptive algorithm [15,16] is developed for a kind of time dependent TKBC.

It is noteworthy to note that SBFEM could be more computationally expensive than FEM, mainly because an eigenvalue problem needs to be solved in generating system equations [17]. Thus, a proof that the appended stiffness matrix is block-circulant is presented if TKBC is cyclically symmetric, resulting in a possible partitioning computation to reduce the computational cost.

In the numerical verification, problems with nonlinear TKBC defined by dual-linear model, time dependent TKBC defined by a viscoelastic model, and cyclically symmetric TKBC are solved.

The paper commences with governing equations of SBFEM with TKBC in Section 2. Section 3 gives descriptions of TKBC. Section 4 proposes numerical algorithms for different kinds of TKBC and presents a proof that the appended stiffness matrix is block-circulant when the TKBC are cyclically symmetric. Section 5 provides a numerical verification via three examples, and the Section 6 summarizes conclusions.

2. Governing equations of SBFEM

The SBFEM introduces a normalised radial coordinate system by scaling the domain boundary relative to a scaling centre (x_0,y_0) selected within the domain, as shown in Fig. 1. The normalised radial coordinate ξ runs from the scaling centre towards the boundary, the other circumferential coordinate s specifies a distance around the boundary from an origin on the boundary.

For two-dimensional elastostatic problems, the SBFEM obtains an approximate solution using the weighted summation of n modes, such that the displacement at any point within a domain is [9]

$$\mathbf{u}(\xi, s) = \mathbf{N}(s)\mathbf{u_h}(\xi) \tag{1}$$

where $\mathbf{N}(s)$ refers to a matrix of circumferential shape functions applying for all lines with a constant ξ. The unknown vector $\mathbf{u}_h(\xi)$ is a set of functions analytical in ξ.

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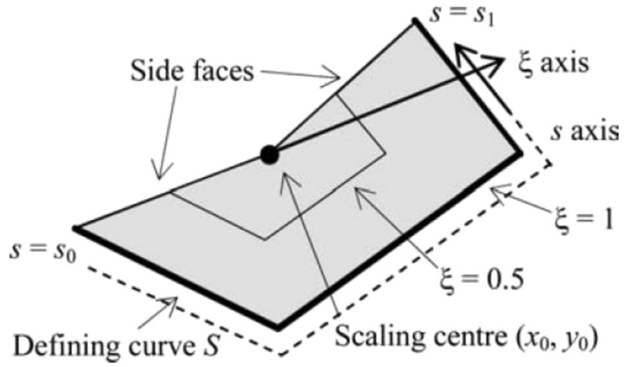

Fig. 1. Scaled boundary coordinate system.

The stresses are obtained by multiplying the strains (obtained from the displacement field using a linear operator) and the elasticity matrix **D** in the form [9]

$$\boldsymbol{\sigma}(\xi,s)=\mathbf{D}\mathbf{B}^1(s)\mathbf{u_h}(\xi)_{,\xi}+\frac{1}{\xi}\mathbf{D}\mathbf{B}^2(s)\mathbf{u_h}(\xi) \tag{2}$$

By the virtue of the virtual displacement principle, one has (in the absence of body force)

$$\int_V \delta\boldsymbol{\varepsilon}(\xi,s)^T\boldsymbol{\sigma}(\xi,s)\mathrm{d}V-\int_s \delta\mathbf{u}(s)^T\mathbf{p}(s)\mathrm{d}s=0 \tag{3}$$

The virtual strain field is of the form

$$\delta\boldsymbol{\varepsilon}(\xi,s)=\mathbf{B}^1(s)\left[\delta\mathbf{u}(\xi)_{,\xi}\right]+\frac{1}{\xi}\mathbf{B}^2(s)[\delta\mathbf{u}(\xi)] \tag{4}$$

where $\delta\mathbf{u}(\xi)$ refers to the vector of virtual displacement.

Utilizing Eq. (1) and substituting Eqs. (2) and (3) into Eq. (4), the virtual work equation becomes

$$\begin{aligned}&\delta\mathbf{u}^T\left(\mathbf{E}^0\mathbf{u_h}(\xi)_{,\xi}+\mathbf{E}^1\mathbf{u_h}(\xi)\right)-\delta\mathbf{u}^T\mathbf{P}\\&-\int_0^1\delta\mathbf{u}(\xi)^T\left[\mathbf{E}^0\xi\mathbf{u_h}(\xi)_{,\xi\xi}+\left(\mathbf{E}^0+\mathbf{E}^{1T}-\mathbf{E}^1\right)\mathbf{u_h}(\xi)_{,\xi}-\mathbf{E}^2\frac{1}{\xi}\mathbf{u_h}(\xi)\right]d\xi\end{aligned} \tag{5}$$

where

$$\begin{aligned}\mathbf{E}_0&=\int_S\mathbf{B}_1(s)^{\mathbf{T}}\mathbf{D}\mathbf{B}_1(s)|\mathbf{J}(s)|\mathrm{d}s\\\mathbf{E}_1&=\int_S\mathbf{B}_2(s)^{\mathbf{T}}\mathbf{D}\mathbf{B}_1(s)|\mathbf{J}(s)|\mathrm{d}s\\\mathbf{E}_2&=\int_S\mathbf{B}_2(s)^{\mathbf{T}}\mathbf{D}\mathbf{B}_2(s)|\mathbf{J}(s)|\mathrm{d}s\end{aligned} \tag{6}$$

where the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are related to the polynomial shape functions [9], **J** is the Jacobian at the boundary ($\xi=1$).

Due to the arbitrariness of $\delta\mathbf{u}(\xi)$, the following conditions must be satisfied

$$\mathbf{P}=\mathbf{E}^0\mathbf{u}_h(\xi)_{,\xi}+\mathbf{E}^{1^T}\mathbf{u}_h(\xi) \tag{7}$$

$$\mathbf{E}^0\xi^2\mathbf{u_h}(\xi)_{,\xi\xi}+\left(\mathbf{E}^0+\mathbf{E}^{1T}-\mathbf{E}^1\right)\xi\mathbf{u_h}(\xi)_{,\xi}-\mathbf{E}^2\mathbf{u_h}(\xi)=\mathbf{0} \tag{8}$$

By inspection, solutions to the homogeneous set of Euler–Cauchy differential equations represented by Eq. (8) must be of the form

$$\mathbf{u_h}(\xi)=c_1\xi^{-\lambda_1}\boldsymbol{\varphi}_1+c_2\xi^{-\lambda_2}\boldsymbol{\varphi}_2+\ldots \tag{9}$$

$\boldsymbol{\phi}_i$ are the independent modals of deformation and λ_i are the modal scaling factors for the 'radial' direction. $\boldsymbol{\phi}_i$ and λ_i are obtained by solving the following eigenproblem [9]

$$\begin{bmatrix}\mathbf{E}_0^{-1}\mathbf{E}_1^T & -\mathbf{E}_0^{-1}\\ \mathbf{E}_1\mathbf{E}_0^{-1}\mathbf{E}_1^T-\mathbf{E}_2 & -\mathbf{E}_1\mathbf{E}_0^{-1}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\varphi}\\\mathbf{q}\end{Bmatrix}=\lambda\begin{Bmatrix}\boldsymbol{\varphi}\\\mathbf{q}\end{Bmatrix} \tag{10}$$

$\mathrm{C}=\{c_1,c_2\ldots\ldots c_i..\}^T$ can be determined at $\xi=1$ via

$$\mathbf{C}=\boldsymbol{\Phi}^{-1}\mathbf{u_h} \tag{11}$$

where $\boldsymbol{\Phi}=[\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\ldots,\boldsymbol{\phi}_n]$

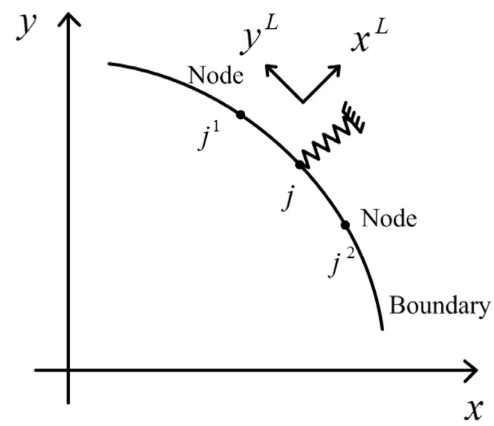

Fig. 2. A boundary with linear TKBC.

Furthermore

$$\mathbf{P}=\mathbf{Q}\mathbf{C}=\mathbf{Q}\boldsymbol{\Phi}^{-1}\mathbf{u_h}=\mathbf{K}\mathbf{u_h} \tag{12}$$

where **P** stands for the vector of nodal force concerned with both the second and third kind boundary conditions, and $\mathbf{Q}=[\mathbf{q}_1,\mathbf{q}_2,\ldots\ldots,\mathbf{q}_n]$, $\mathbf{q}_l$.refers to the dual variable of $\boldsymbol{\phi}_l$.

The nodal forces concerned with TKBC is described by $\mathbf{P_S}$

3. Description of the third kind boundary condition in static problems

The third kind boundary condition is employed to describe a relationship between variables and their derivatives along the boundary. For 2-D static problem, it can be written as

$$\begin{bmatrix}l_1 & 0 & l_2\\ 0 & l_1 & l_2\end{bmatrix}\boldsymbol{\sigma}=\mathrm{p_s}(\mathbf{u})\quad x_i\in\Gamma_\sigma \tag{13}$$

where l_i is the vector of unit outside normal, $\boldsymbol{\sigma}$ stands for the vector of stress, and relates the derivatives of **u** with constitutive relationships, **u** designates the vector of displacement. $\mathbf{p_s}$ refers to the vector of traction, and is a function of **u**, which may be linear, nonlinear, and time dependent, explicitly or implicitly.

Assume the relationship between $\mathbf{p_s}$ and **u** is nonlinear, and is defined by

$$\mathbf{p}_s=\mathrm{A}(\mathbf{u})\mathbf{u} \tag{14}$$

where **A(u)** is a matrix concerned with u.

Assume the relationship between $\mathrm{p_s}$ and **u** is time dependent, and is defined by

$$\mathbf{p}_s(t)=\Lambda\mathbf{u} \tag{15}$$

where Λ refers to a matrix of temporal operators.

In the SBFEM, the boundary is discretized into a number of elements (shown in Fig. 2), the displacement $\mathbf{u}_j=(u_{jx},u_{jy})^T$ at the jth element is approximated by

$$\mathbf{u}_j=\mathbf{N}(s)\mathbf{u}_j^e \tag{16}$$

where $\mathbf{u}_j^e$ stands for the vector of nodal displacements of jth element.

The equivalent nodal forces concerned with TKBC at the jth element can be written as

$$\mathbf{P}_{\mathbf{S}_j^e}=\int_\Gamma \upsilon(s)\mathbf{N}^T(s)\mathbf{p}_{\mathbf{s}j}(s)d\Gamma \tag{17}$$

where

$$\upsilon=\begin{cases}1 & ,\quad \text{Distributive traction}\\ \delta(s-s_j) & ,\quad \text{Individualload}\end{cases} \tag{18}$$

where $\mathbf{p}_{\mathbf{s}j}(s)$ designates a vector of traction at the jth element, and δ is Dirac function.