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The method of transformed angular basis function for solving the Laplace equation

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ABSTRACT

In this paper, we propose a new approach to improve the method of angular basis function (MABF) proposed by Young et al. (2015) for the Laplace equation in two-dimensional settings. Instead of the fundamental solution $\ln r$ used in the traditional Method of Fundamental Solution (MFS), MABF employs a different basis function θ and produces good approximate solutions on the domains with acute, narrow regions and exterior problems (Young et al., 2015). However, the definition of θ inevitably incurs a singularity situation for many different types of domains. Therefore, the selection of source points of MABF is not as convenient as the traditional MFS. To avoid the singularity situation in implementing, we introduce a transformation so that the transformed angular basis function does not exhibit this type of singularity for commonly used distributions of source points. As a result, source points for the method of transformed angular basis function (MTABF) can then be chosen in a similar way to traditional MFS. Numerical experiments demonstrate that the proposed approach significantly simplifies the selection of source points in MABF for different types of domains, which makes MABF more applicable. Numerical results of MTABF and MFS are presented for comparison purposes.

1. Introduction

The method of fundamental solution (MFS) was originally introduced by Kupradze and Aleksidze [11]. The implementation of MFS was studied for the first time by Mathon and Johnston [15]. MFS approximates the solution of the problem by a linear combination of fundamental solutions over a discrete set of source points placed outside of the domain. Therefore, the coefficients of the MFS approximation are determined by solving a linear problem. In the past two decades, MFS has attracted a lot of attentions from science and engineering community [2,5,8,10,12–14,16]. One of the main advantage of MFS is that it voids the complex mesh generations and numerical integrations. Survey papers of the MFS and related methods can be found in [4,6,7,9].

It is well-known that traditional method of fundamental solution (MFS) adopts the fundamental solution $\ln r$ of the 2-D Laplace equation. Actually, the function $\ln r$, as a function of the radial variable, is the real part of the complex fundamental solution of the Laplace equation. Through the complex variable theorem, the solution could be fully expressed in terms of radius and argument (see [3,17]). Furthermore, the imaginary part discussed in [3] can be simplified as a function of argument satisfying the Laplace equation when the source point is taken at the origin. This simplified function was called an angular basis function and has been used to construct the method of angular basis functions (MABF) in [17]. Therefore, MABF studied in [17] can be viewed as the MFS using this angular basis function, which is different from the traditional MFS using $\ln r$. MABF has been numerically shown to be a good substitute for the traditional method of fundamental solution (MFS) in solving the Laplace equation. However, to determine the locations of source points for this method is not straightforward. The authors of [17] proposed a distribution of source points to avoid any pair of a collocation point and a source points resting on a horizontal line so that the angular basis function $\theta(x, y)$ can be well defined. Therefore, there remains a crucial question of developing a simple approach for the selection of source points for MABF. Another study involving angular-type fundamental solution has been reported in [3] the Trefftz methods by using degenerate kernels and Fourier series to formulate the angular-type fundamental solution and then to successfully solve an infinite domain with circular holes and/or inclusions subject to a screw dislocation.

The aim of this paper is to develop an algorithm for MABF so that source points can be easily chosen as in MFS that uses radial basis function $\ln r$. Toward this end, we propose a transformation for angular basis function θ to avoid possible singular situations. The implementation of this transformation proceeds in three steps. Firstly, source points are

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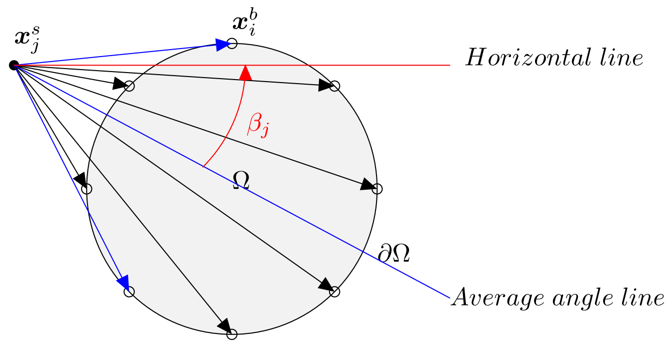

Fig. 1. The angle β_j for the transformation from Θ_{ij} to $\overline{\Theta}_{ij}$. Solid dot: a source point $\boldsymbol{x}_j^s$; open circles: boundary collocation points $\boldsymbol{x}_i^b$ for all $i = 1, 2, \ldots, N$.

placed surrounding the domain of the problem such that any source point should not stay interior of a convex region that contains all collocation points. Secondly, we calculate the average angle of all vectors pointing from a source point toward all collocation points. Then, we rotate all vectors about this source point through the average angle counterclockwise so that all angles are distributed quite evenly on $(-\pi/2, \pi/2)$. This process can be done by multiplying a rotation matrix generated by the average angle to each vector. As a result, all vectors are transformed to their reference positions (see Fig. 1). Thirdly, a transformed angular basis function (TABF) is then defined to be the direction angle of a transformed vector, which falls into $(-\pi/2, \pi/2)$. It can be verified that a TABF is also a fundamental solution of Laplace equation. The calculation of these TABFs on a set of source points results in the method of transformed angular basis function (MTABF), which can be seen as an improvement of MABF. Numerical experiments show that the effort in selecting source points is substantially reduced for different types of domains. Actually, many source point distributions that are commonly used in MFS also work for MTABF.

The rest of the paper is organized as follows: in Section 2 we introduce transformed angular basis functions. Formulation of MTABF is presented in Section 3. Numerical results and comparison are presented in Section 4. We end by some concluding remarks in Section 5.

2. Transformed angular basis functions

Let Ω be a bounded domain. Its boundary is denoted by ∂Ω. Let

$$\boldsymbol{X}^b := \{\boldsymbol{x}_i^b = (x_i^b, y_i^b), i = 1 \ldots N\}$$

be a set of collocation points on ∂Ω, and

$$\boldsymbol{X}^s := \{\boldsymbol{x}_j^s = (x_j^s, y_j^s), j = 1 \ldots N\}$$

be a set of source points on the boundary of a convex region containing Ω. Here, N denotes the number of collocation points. In this paper, we use the same number of source points as collocation points.

Furthermore, we denote by X and Y the distance matrices for variable x and y, respectively, i.e.,

$$X = \left(X_{ij}\right) \quad \text{and} \quad Y = \left(Y_{ij}\right), \tag{2.1}$$

where

$$X_{ij} := x_i^b - x_j^s, \quad \text{and} \quad Y_{ij} := y_i^b - y_j^s, \tag{2.2}$$

for any $i, j = 1 \ldots N$.

We now calculate the angle of each vector pointing from a source point $\boldsymbol{x}_j^s$ toward a boundary collocation point $\boldsymbol{x}_i^b$ with respect to x-axis. These angles can be written into the following original angular matrix

$$\Theta := \arctan\left(\frac{Y}{X}\right) = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \ldots & \Theta_{1N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Theta_{N1} & \Theta_{N2} & \ldots & \Theta_{NN} \end{pmatrix} \tag{2.3}$$

Here each entry Θ_{ij} will be calculated by

$$\Theta_{ij} := \arctan\left(\frac{Y_{ij}}{X_{ij}}\right).$$

Note that function $\theta(x, y) := \arctan \frac{y}{x}$, that is known as the angular basis function (see [17]), can be used to construct the expression of the argument of complex number $z = x + iy$, which is usually denoted by $arg(z)$. A branch cut, usually along the negative real axis, can limit $arg(z)$ so it lies between $(-\pi, \pi]$. As pointed out in [17], the value of Θ_{ij} obtained from commonly used source point distributions would result in an ill-conditioning linear system. Therefore, the location of source points must be carefully determined. In order to alleviate the difficulty in choosing source points, we will design a transformation for Θ_{ij} so that source points can be selected in an easy way as for MFS. To illustrate this idea, we consider a disk domain Ω (see Fig. 1). We plot vectors $\overrightarrow{\boldsymbol{x}_j^s \boldsymbol{x}_i^b}$ $(i = 1, 2, \ldots, N)$ from a given source point $\boldsymbol{x}_j^s$ toward all collocation points $\boldsymbol{x}_i^b$ on the boundary ∂Ω.

We first find the average angle line by averaging the maximal and minimal angles among all angles of these vectors. Then an angle $\beta \in [0, 2\pi)$ between the average angle line and the horizonal line can be figured out. Secondly, we transform all vectors by using a rotation matrix with angle β_j so that angles of these transformed vectors fall into the interval $(-\pi/2, \pi/2)$. This transformation can be designed for any given domain if the locations of sources points are properly selected.

2.1. Calculation of β

Next, we will give a detailed description about the calculation of angle β_j that labeled in Fig. 1. Without loss of generality, we work out β_j for vectors $\overrightarrow{\boldsymbol{x}_j^s \boldsymbol{x}_i^b}$ $(i = 1, 2, \ldots, N)$. The β values for vectors starting from other source points can be found in a similar manner. Actually, we have

$$\beta_j = \begin{cases} \pi - \frac{1}{2}(\gamma_1 + \gamma_2), & \text{if} \quad \gamma_3 - \gamma_4 > \pi \ \text{ and } \ \gamma_3 + \gamma_4 \le 2\pi, \\ 3\pi - \frac{1}{2}(\gamma_1 + \gamma_2), & \text{if} \quad \gamma_3 - \gamma_4 > \pi \ \text{ and } \ \gamma_3 + \gamma_4 > 2\pi, \\ 2\pi - \frac{1}{2}(\gamma_3 + \gamma_4), & \text{otherwise,} \end{cases} \tag{2.4}$$

where

$$\gamma_1 := \max\{\vartheta \in \{\theta_{ij}\}_{i=1}^N \mid \vartheta < \pi\}, \qquad \gamma_2 := \min\{\vartheta \in \{\theta_{ij}\}_{i=1}^N \mid \vartheta > \pi\},$$
$$\gamma_3 := \max\{\theta_{ij}\}_{i=1}^N, \qquad \gamma_4 := \min\{\theta_{ij}\}_{i=1}^N,$$
$$\theta_{ij} := arg(z_{ij}), \qquad z_{ij} := (x_i^b - x_j^s) + \boldsymbol{i}(y_i^b - y_j^s).$$

Multiplying the rotation matrix on each vector $\overrightarrow{\boldsymbol{x}_j^s \boldsymbol{x}_i^b}$ we obtain

$$\begin{bmatrix} \overline{X}_{ij} \\ \overline{Y}_{ij} \end{bmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{bmatrix} X_{ij} \\ Y_{ij} \end{bmatrix}. \tag{2.5}$$

As a consequence, each entry of matrix Θ can be transformed into a new matrix $\overline{\Theta}$ as follows

$$\overline{\Theta} := \arctan\left(\frac{\overline{Y}}{\overline{X}}\right) = \left(\arctan\left(\frac{\overline{Y}_{ij}}{\overline{X}_{ij}}\right)\right)_{N\times N}. \tag{2.6}$$

We already complete the design of the transformation for the angle matrix Θ. Next, we will consider the transformation of the angular basis function.

2.2. The ABF $\phi(x, y) = \theta$ and the corresponding TABF $\varphi(x, y) = \overline{\theta}$

Consider the angular basis function (see [17]):

$$\phi(x, y) = \theta = \arctan\left(\frac{Y}{X}\right),$$

where $\langle X, Y\rangle = \langle x - x_0, y - y_0\rangle$ denotes a vector in xy-plane starting at (x_0, y_0). It is straightforward to verify that $\theta(x, y)$ is a fundamental solution of the Laplace equation. We define the transformed angular basis