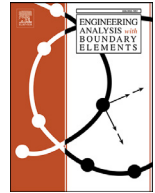




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Meshless analysis of elliptic interface boundary value problems

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ABSTRACT

In the present paper, Multiquadric radial basis function (MQ RBF) and its integrated form are used to construct collocation methods for numerical solution of two-dimensional elliptic problems with curved or closed interface. The main purpose of this work is to perform a comparative analysis of both the methods via accuracy and condition number of the coefficient matrix for elliptic interface problems. In the classical RBF collocation method, the shape parameter is selected by using cross validation approach [1]. In the case of Integrated MQ RBF, a reasonable accuracy is obtained for a wide range of values of the shape parameter. Some of the benchmark problems such as linearized Poisson-Boltzmann problem [2], Poisson interface problem [3], Pennes Bioheat Equation [4] (with no exact solution, containing two phases), are considered to validate accuracy and efficiency of the RBFs collocation methods.

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1. Introduction

Meshless methods have been used in science and engineering in the context of numerical solution of Partial Differential Equations (PDEs). The major distinctions of the meshless methods are elimination of mesh generation and better suitability for complex geometries [4]. On the contrary, mesh-based numerical methods such as finite-difference method, finite-volume method and finite-element method become more complex in higher dimension. However, in spite of these advantages, meshless methods have also some drawbacks. For instance, meshless methods based on shape dependent radial basis functions (RBFs), such as multiquadric, inverse multiquadric, inverse quadric and Gaussian have strong connection between condition number (which is unfortunately high in most cases) of the matrix and accuracy of the method.

The condition number of the matrix has direct bearings on the numerical stability of the approximation method. So suitable selection of the shape parameter ϵ is very important to accuracy of the numerical results. It has been observed in different experiments that small values of shape parameter ϵ leads to accurate numerical results, but at the same time is responsible for ill-conditioning of the system matrix. Due to ill-conditioning, the accuracy is reversed and the numerical method often becomes useless. Different techniques for the selection of relatively optimal value of shape parameter ϵ are shown in [1,5–11]. However, finding a valid and theoretically sound procedure of selecting the optimal shape parameter ϵ is still an open problem.

Interface problems arise in various branches of sciences and engineering, such as biological systems and material science [12]. Elliptic

interface problem also have wide applications in fluid mechanics [13] and electromagnetic wave propagation [14–17]. The Poisson-Boltzmann equation and its linearized form is used to predict the electrostatic effects for biomolecular systems. This popular model is used for describing electrostatic interactions between molecules in ionic solutions [18]. Poisson equation with discontinuities have basic importance in the description of fluid flows separated by interfaces like the contact surfaces for immiscible multiphase fluids or fluids separated by a membrane [19].

Due to importance of interface problems, a number of methods are reported in literature. Among them are the immersed boundary method [20,21], the immersed interface method [22,23] and the ghost fluid method [24,25]. Finite element methods for the interface problems include the work [2,26–28] and the references therein. Spectral and radial basis function based methods for one-dimensional elliptic interface problems is discussed in [29]. In [3], a radial basis collocation method is applied to investigate two-dimensional heterogeneous conduction and bioheat transfer problem with interface condition.

The main contributions of the current works is to explore further application of the Kansa's collocation radial basis functions method [30,31] and its modified form (meshless collocation method based on integrated MQ RBF) for the numerical solution of two-dimensional elliptic PDEs involving s-type of curved, square, circular and star-shaped interfaces. Accuracy and efficiency of these methods are thoroughly investigated and the results are compared with DMLPG [2] having homogeneous and non-homogeneous jump in the solution and its derivative across the interface. Furthermore, these methods are used for the numerical solution of Pennes bioheat equation, whose exact solution is not

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available. The overall performance of the integrated MQ RBF is found better than the other companion methods. Moreover, dependency of accuracy of the meshless method based on integrated MQ RBF on the shape parameter ϵ is undertaken as well.

2. Governing equation

We consider a two-dimensional elliptic equation [2] of the form,

$$\mathcal{L}u(\mathbf{x}) := \nabla \cdot (\beta(\mathbf{x})\nabla u(\mathbf{x})) + \sigma(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \Omega, \quad (1)$$

where ∇ is the divergence operator and Dirichlet boundary condition is assumed on the boundary $\partial\Omega$ of the solution domain Ω . i.e.,

$$Bu(\mathbf{x}) := u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \quad (2)$$

The solution of (1) is discontinuous across the interface Γ in the given solution domain Ω . The interface Γ divides the solution domain Ω into two disjoint sub-domains. The notations Ω^+ and Ω^- are used to represent the disjoint sub-domains, where the notation $u^+(\mathbf{x})$ and $u^-(\mathbf{x})$ are used to represent the solution in each of the respective sub-domains. In addition, two more conditions are required on the interface Γ to solve the PDE (1) with given boundary conditions. These are the jump conditions which are specified along the interface Γ as,

$$\begin{aligned} u^+(\mathbf{x}) - u^-(\mathbf{x}) &= w(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \\ \beta^+(\mathbf{x}) \frac{\partial}{\partial n} u^+(\mathbf{x}) - \beta^-(\mathbf{x}) \frac{\partial}{\partial n} u^-(\mathbf{x}) &= v(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \end{aligned} \quad (3)$$

where n is the unit normal direction. The functions $f(\mathbf{x})$, $g(\mathbf{x})$, $\beta(\mathbf{x})$, $\sigma(\mathbf{x})$, $w(\mathbf{x})$ and $v(\mathbf{x})$ are the known functions. The two functions $w(\mathbf{x})$ and $v(\mathbf{x})$ defined on the Γ represent discontinuities across the interface Γ . The function $w(\mathbf{x})$ is used for the jump in the function values and $v(\mathbf{x})$ is used for the jump in the normal derivatives. The coefficients $\beta(\mathbf{x})$ and $\sigma(\mathbf{x})$ can also be discontinuous across the interface Γ , i.e.,

$$\beta(\mathbf{x}) = \begin{cases} \beta^+(\mathbf{x}) & \mathbf{x} \in \Omega^+, \\ \beta^-(\mathbf{x}) & \mathbf{x} \in \Omega^-, \end{cases}$$

$$\psi_j(\mathbf{x}) = \frac{\sqrt{1 + (\epsilon r)^2} \{40(\epsilon r)^6 - 1518(\epsilon r)^4 + 1779(\epsilon r)^2 - 128\} + 105\epsilon r \sinh^{-1}(\epsilon r) \{8(\epsilon r)^4 - 20(\epsilon r)^2 + 5\}}{201600\epsilon^6}, \quad (9)$$

$$\sigma(\mathbf{x}) = \begin{cases} \sigma^+(\mathbf{x}) & \mathbf{x} \in \Omega^+, \\ \sigma^-(\mathbf{x}) & \mathbf{x} \in \Omega^-. \end{cases}$$

The governing equation (1) is known as linearized Poisson-Boltzmann equation in the case of weak electrostatic potential, f and u are the scaled singular charge distribution and electrostatic potential respectively [32].

3. Meshless method

The meshless solution of (1) is calculated separately in each sub-domain at collocation points, which can be written as

$$\hat{u}(\mathbf{x}) = \begin{cases} u^+(\mathbf{x}) = \lambda_1^+ \phi_1^+(\mathbf{x}) + \lambda_2^+ \phi_2^+(\mathbf{x}) + \dots + \lambda_{N_s^+}^+ \phi_{N_s^+}^+(\mathbf{x}), & \mathbf{x} \in \Omega^+, \\ u^-(\mathbf{x}) = \lambda_1^- \phi_1^-(\mathbf{x}) + \lambda_2^- \phi_2^-(\mathbf{x}) + \dots + \lambda_{N_s^-}^- \phi_{N_s^-}^-(\mathbf{x}), & \mathbf{x} \in \Omega^-, \end{cases} \quad (4)$$

where N_s^- and N_s^+ are the total number of source points in the sub-domains Ω^- and Ω^+ , respectively. The functions

$$\begin{aligned} \phi_j^+(\mathbf{x}) &= \phi^+(\|\mathbf{x} - \mathbf{x}_j\|_2), \quad j = 1, 2, 3, \dots, N_s^+, \\ \phi_j^-(\mathbf{x}) &= \phi^-(\|\mathbf{x} - \mathbf{x}_j\|_2), \quad j = 1, 2, 3, \dots, N_s^-, \end{aligned} \quad (5)$$

are the given RBFs. For solution of the governing equation (1), the coefficients λ_j^+ and λ_j^- need to be evaluated at all collocation points. To obtain these coefficients we substitute (4) into (1)–(3), which results in a system of linear equations to be solved for both λ_j^+ and λ_j^- .

3.1. Identification of collocation points

First step of the RBF-based meshless method is to identify the boundaries and the interface for each sub-domain. A set of scattered nodes is selected in the computational domain Ω as

$$\hat{\Omega} = \{\mathbf{x}_k\}_{k=1}^M, \quad \mathbf{x}_k = (x_k, y_k),$$

which consist of four disjoint subsets representing the sub-domains, the interface Γ and its boundary part $\partial\Omega$ (see Fig. (1)) i.e.,

$$\begin{aligned} \hat{\Omega}^+ &= \{\mathbf{x} \in \hat{\Omega} \cap \Omega^+\} \subset \Omega^+, \\ \hat{\Omega}^- &= \{\mathbf{x} \in \hat{\Omega} \cap \Omega^-\} \subset \Omega^-, \\ \partial\hat{\Omega} &= \{\mathbf{x} \in \hat{\Omega} \cap \partial\Omega\} \subset \partial\Omega, \\ \hat{\Gamma} &= \{\mathbf{x} \in \hat{\Omega} \cap \Gamma\} \subset \Gamma. \end{aligned} \quad (6)$$

We assume that $\hat{\Omega} = \hat{\Omega}^+ \cup \hat{\Omega}^- \cup \partial\hat{\Omega} \cup \hat{\Gamma}$ and none of these subset is empty. The $\hat{\Omega}^+$ represents the set of discrete collocation points in the sub-domain Ω^+ , $\hat{\Omega}^-$ represents a set of discrete collocation points in the sub-domain Ω^- . The collection $\partial\hat{\Omega}$ is a set of discrete collocation points on the boundary $\partial\Omega$, $\hat{\Gamma}$ represents a set of discrete collocation points on the interface Γ . We further assume that the total number of collocation points in Ω^+ is N^+ and in Ω^- is N^- . Similarly, the total number of collocation points in $\partial\Omega$ is K^+ and total number of collocation points in interface Γ is K^- .

3.2. Radial basis functions

In the present work, two types of RBFs will be used. i.e.,

$$\phi_j(\mathbf{x}) = \begin{cases} \varphi_j(\mathbf{x}), & \text{in case of MQ RBF,} \\ \psi_j(\mathbf{x}), & \text{in case of Integrated MQ RBF,} \end{cases} \quad (7)$$

where

$$\varphi_j(\mathbf{x}) = \sqrt{1 + c^2 r^2}, \quad (8)$$

$$r = \sqrt{(x - x_j)^2 + (y - y_j)^2}.$$

It is reported in [31,33] that Integrated MQ RBF has better approximating properties than the ordinary MQ RBF. In this work we consider the ordinary MQ RBF (8) and its integrated form (9), which is obtained by six times integration of the MQ RBF. The shape parameters c and ϵ control shapes of the functions i.e., small value of the shape parameters lead to flat RBF, while a large value of the shape parameters lead to more peaked RBF [4,33].

For onwards representation, we introduce the following notations

$$\begin{aligned} \Phi^{+T}(\xi) &= [\phi_1^+(\xi), \phi_2^+(\xi), \phi_3^+(\xi), \dots, \phi_{N_s^+}^+(\xi)], \\ \xi \in \Omega^+, \lambda^+ &= [\lambda_1^+, \lambda_2^+, \lambda_3^+, \dots, \lambda_{N_s^+}^+]^T, \\ \Phi^{-T}(\eta) &= [\phi_1^-(\eta), \phi_2^-(\eta), \phi_3^-(\eta), \dots, \phi_{N_s^-}^-(\eta)], \eta \in \Omega^-, \\ \lambda^- &= [\lambda_1^-, \lambda_2^-, \lambda_3^-, \dots, \lambda_{N_s^-}^-]^T, \end{aligned}$$

where the superscript T represents transpose. Accordingly, (4) can be written in matrix notation as

$$\hat{u}(\mathbf{x}_i) = \begin{cases} \Phi^{+T}(\mathbf{x}_i)\lambda^+, & \text{when } \mathbf{x}_i = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N^+}]^T \in \Omega^+, \\ \Phi^{-T}(\mathbf{x}_i)\lambda^-, & \text{when } \mathbf{x}_i = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N^-}]^T \in \Omega^-, \end{cases} \quad (10)$$

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