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Performance of BEM superposition technique for solving sectorially heterogeneous Laplace's problems with non-regular geometry

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ABSTRACT

The superposition technique is a new BEM alternative for solving sectorially heterogeneous problems in which the complete domain is divided in a surrounding homogeneous domain and other complementary sub-domains with different constitutive properties. It is an alternative to the classic BEM sub-regions technique. Results of preliminary simple problems governed by the Laplace's equation were successfully solved, using analytical solutions to performance evaluation. Thus, this paper examines the performance of the superposition technique to solve complex problems that present geometric irregularities on the boundary, such as grooves and notches, and internal inclusions. Considering the absence of analytical solutions, the Finite Element Method was used to generate the reference solutions for a suitable comparison.

1. Introduction

The great advances of modern engineering result in increasingly sophisticated numerical models, which require the aid of additional mathematical tools even considering the most powerful discrete techniques as Finite Element Method (FEM), Finite Volume Method (FVM) and Boundary Element Method (BEM). One can mention the use of radial basis functions [8], the wavelet techniques [1,2] and the multiscale modeling [10] as examples of these auxiliary approaches. The first two are effective mainly in the implementation of adaptive procedures [13,22] and the third is suitable to describe the behavior of highly heterogeneous materials [9].

The radial basis function is still used successfully to model body forces, sources, advective and inertial effects within the context of the modern BEM techniques as the Dual Reciprocity [20] and the Direct Interpolation [15,18]. These techniques are also capable to gradually simulate heterogeneous materials. However, certain problems remain challenging to the BEM as the agile solution of piecewise or sectorial heterogeneous problems.

Numerical solution of problems with heterogeneous sectors is preferably performed by methods that discretize the domain, such as the mentioned FEM and the FVM, since different values of the properties are easily introduced inside each sector or sub-domain. Using the Boundary Element Method (BEM), computation of sectorial heterogeneities is not immediate and during many years the Sub-region Technique (SRT) has been the more efficient approach to solve this kind of problem [5].

Proposed improvements in SRT, focusing the solution of heterogeneous problems are not numerous. Some works examines the SRT aiming applications in narrow or long profile problems, since the domain can be partitioned and the interaction between source points and field points located distant each other are avoided. For example, Dual reciprocity method [20] in multi domains subdivides the domain into sub-regions preventing computational effort and producing better convergence and better approximation to physical variables, such as variable velocities in diffusive–advective problems [23]. Kita and Kamiya [12] proposed an improvement to the classic sub-region technique in which a global matrix is constructed and better results are achieved; however, the BEM equations should be transformed into equations in form similar to the stiffness equations of FEM. Using this previous idea but examining multi-layer elastic crack problems, Lu and Wu [19] presented a parallelization technique to assemble BEM sub-region matrices that focuses on a reduction in the computational cost. On the other hand, some similar techniques are based on the connection of different regions by internal points. Problems with interaction between soil and structure, employing the FEM and the BEM together [26], as well as cases of plate-beam-column integrated systems [21,25], employ the idea of connection between different regions using common nodal points. Recently, Wagdy and Rashed [27] also proposed an alternative formulation in which the idea of the linkage by internal points is improved.

In a previous work, Brebbia et al. [6] have presented an alternative BEM approach for this important class of problems with relative simplicity, named as Domain Superposition Technique (DST). Using the DST,

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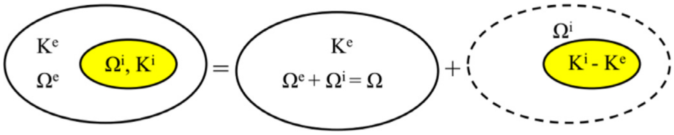

Fig. 1. Surrounding and sectorial domains.

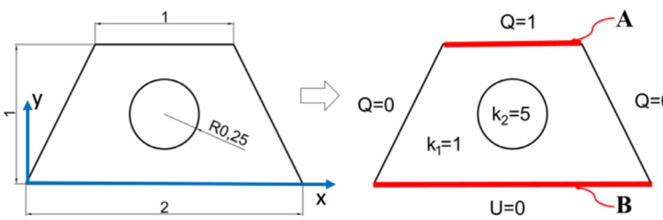

Fig. 2. Geometric features and boundary conditions to the first example.

Table 1
Number of nodes and elements for the BEM and FEM in the first example.

	Elements	Total	Boundary	Inside	In A	in B
FEM	9742	5003	262	4741	51	100
	12,640	6468	294	6174	57	112
	38,594	19,560	524	19,036	101	200
BEM		74	50	24	11	21
		140	100	40	21	41
		270	200	70	41	81

the complete problem is modeled as a superposition of one surrounding homogeneous domain and a set of complementary internal sub-domains with different properties. The energy of each sub-domain is computed to the system as a whole by superposition, similar to what is done with sources or body actions for solving Poisson's problems [14,20]. All sectors are mathematically connected by means of the influence coefficients, which in the BEM standard procedure are generated by integrations performed on the boundary sub-domains, with the source points located at each nodal point generated by the discretization, both external and internal as well.

Initially, relatively simple problems with boundaries and internal sectors with rectangular shape were solved and comparisons with sub-regions technique were carried out [16,17]. Results achieved with the Finite Element Method using finer meshes were used as reference for the performance evaluation. The proposed technique was successful, since its results were always superior to the results obtained by the sub-region technique.

In continuity to the previous works, more elaborate applications are presented here, in which the shape of two-dimensional domains presents grooves, notches and other geometric irregularities. Internal inclusions that characterize the heterogeneities also have non-regular shapes. Applications are done in stationary problems governed by the Laplace equation. In the absence of analytical solutions, the performance comparison in each case is carried out considering the results obtained by the Finite Element Method [4,11] with a finer mesh.

2. Domain superposition formulation

Consider a domain consisting of two regions with different physical properties. Within each sub-domain these properties are constant, as shown in Fig. 1. Thus, the complete domain $\Omega(\mathbf{X})$ is composed of the sum of the sub-domains $\Omega^e(\mathbf{X})$ and $\Omega^i(\mathbf{X})$, with constitutive properties respectively given by K^e and K^i:

Assuming that $K^s = K^e + K^i$, the following integral equation can be written:

$$\int_\Omega K(\mathbf{X})u(\mathbf{X})_{,ii}u*(\xi;\mathbf{X})d\Omega(\mathbf{X})$$
$$= K^e\int_\Omega u(X)_{,ii}u*(\xi;\mathbf{X})d\Omega(\mathbf{X}) + K^s\int_{\Omega^i} u^i(X)_{,ii}u*(\xi;\mathbf{X})d\Omega^i(\mathbf{X}) = 0 \quad (1)$$

In Eq. (1), $u^*(\xi{:}\mathbf{X})$ is the fundamental solution [3] and $u^i(\mathbf{X})$ means the potential in internal points. Rewriting the second term on the right side of Eq. (1) in a boundary integral form [16], highlighting that the source points are also located on the domain $\Omega^i(\mathbf{X})$ for clarity, one has:

$$K^s\int_{\Omega^i} u^i(X)_{,ii}u*(\xi;\mathbf{X})d\Omega^i(\mathbf{X})$$
$$= K^s\left[\int_{\Gamma^i} q^i(\mathbf{X})u*(\xi;\mathbf{X})d\Gamma^i(\mathbf{X}) - \int_{\Gamma^i} u^i(\mathbf{X})q*(\xi;\mathbf{X})d\Gamma^i(\mathbf{X}) - c(\xi)u^i(\xi)\right] \quad (2)$$

In Eq. (2), $q^*(\xi{:}\mathbf{X})$ is the normal derivative of the fundamental solution and $q^i(\mathbf{X})$ are the normal derivatives of potentials on internal boundaries. The value of $c(\xi)$ in Eq. (2) is dependent on the position of the source point in relation to the boundary and, if it is located on it, its smoothness [16].

Physically, the BEM integral equation for Laplace's problem is related to the energy balance in the system, considering the balance between the diffusive energy and the flux work. In a homogeneous case, these energies are computed respectively by potentials and normal derivatives values on the boundary. Considering the DST strategy, the sectorial heterogeneities have an intrinsic energy that should be also computed in the total energy of system. Thus, the first boundary integral on the right-hand side of Eq. (2) represents the flux work $q^i(\mathbf{X})$, while the other terms represent the diffusive energy, which is expressed as a function of the internal potentials.

The aim of the proposed method is consider only the evaluation of the amount of diffusive energy present in the internal sub-domains, such as is done in Poisson's problems computing the work due to a source or external action [7].

Considering the diffusive energy in its totality, which is much easier to compute since it is given as a function of the potentials at the inner points $u^i(\mathbf{X})$, the two integrals on the right hand side of Eq. (1) are rewritten as:

$$\int_\Gamma u(\mathbf{X})q*(\xi;\mathbf{X})d\Gamma(\mathbf{X}) - \int_\Gamma q(\mathbf{X})u*(\xi;\mathbf{X})d\Gamma(\mathbf{X}) + c(\xi)u(\xi)$$
$$= -\frac{K^s}{K^e}\left[c(\xi)u^i(\xi) + \int_{\Gamma^i} u^i(\mathbf{X})q*(\xi;\mathbf{X})d\Gamma^i(\mathbf{X})\right] \quad (3)$$

As mentioned, the right hand side of Eq. (3) has the same meaning of a domain integral that computes the work due to a source applied to a sector for the Poisson's problems. However, here, this energy is given directly by a boundary integral and the internal potentials are unknown.

It is highlighted that the work of flows in the internal domain is not zero; however, it is enough to compute the total diffusive energy of the sub-domain in the energy balance [17]. The entire left hand side of Eq. (3) is affected by the complete diffusive energy introduced. Thus, the system responds as a whole, i.e., the potential and normal derivative calculated by the final system of equations take into account the effect of all sub-domains and the surrounding domain as well.

Since the energy of each partition is represented exclusively by means of potential values at the internal points, these points must appear explicitly in the matrix system, that is, the potential associated with them are calculated simultaneously with boundary nodal points. Then, the BEM matrix system after the discretization procedure is thus given by:

$$\begin{pmatrix}\mathbf{H}_{cc} & \mathbf{H}_{ci}\\ \mathbf{H}_{ic} & \mathbf{H}_{ii}\end{pmatrix}\begin{pmatrix}\mathbf{u}_c\\ \mathbf{u}_i\end{pmatrix} = \begin{pmatrix}\mathbf{G}_{cc} & \mathbf{0}_{ci}\\ \mathbf{G}_{ic} & \mathbf{0}_{ii}\end{pmatrix}\begin{pmatrix}\mathbf{q}_c\\ \mathbf{q}_i\end{pmatrix} \quad (4)$$