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A meshfree method with plane waves for elastic wave propagation problems

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ABSTRACT

In this paper, we address the meshfree numerical solution of time-harmonic linear elastic wave propagation problems in homogeneous media. In particular, we analyze the asymptotic behavior of the method of fundamental solutions (MFS) with source points located far away from the domain of interest. The asymptotic MFS is shown to be equivalent to a Trefftz method, here referred to as the plane waves method (PWM), based on superposition of shear and compressional elastic plane waves with different directions of propagation. Several numerical examples are included in order to illustrate the equivalence between the asymptotic MFS and the PWM. The convergence and stability of the PWM are also analyzed in smooth settings.

1. Introduction

The method of fundamental solutions (MFS) [1–4] is a meshfree and integration free boundary collocation technique that falls into the class of Trefftz methods [5]. In the MFS, the unknown solution of a homogeneous elliptic boundary value problem (BVP) is approximated by superposition of fundamental solutions of the corresponding differential operator, with singularities distributed on a pseudo boundary [6] located in the exterior of the domain.

According to the reported numerical results [7–9], the accuracy of the MFS, when applied in smooth settings, may be improved by increasing the distance between the domain of interest and the pseudo boundary containing the singularities of the shape functions. However, this improvement is limited by the machine precision and the error accumulation due to the ill-conditioning of the corresponding linear systems. In this sense, it is necessary to analyze how far away can we push the singularities in order to achieve accurate numerical results, without destroying the stability of the method.

Besides on the number (and location) of singularities and boundary collocation points, the optimal MFS accuracy depends on the shape of the domain and on the regularity of the boundary conditions, e.g. [9]. In other words, the highest accuracy of the MFS does not necessarily correspond to taking the most distant singularities. The goal of this paper is to analyze the asymptotic behavior of the MFS for far away singularities and not to establish an optimality condition for the MFS accuracy.

The behavior of the classical MFS with source points distributed on a spherical pseudo boundary with an arbitrarily large radius was analyzed in [10,11], for interior acoustic wave propagation problems. In the referred publications it was shown that this asymptotic case of the MFS is equivalent to a meshfree method, called the plane waves method (PWM), based on superposition of acoustic plane waves with different directions of propagation.

Since its introduction, the PWM has been successfully applied for the numerical calculation of eigenfrequencies and eigenmodes of 2D and 3D domains [12] and for estimating eigenmodes for acoustic cavities [13,14]. Also, inverse problems for Helmholtz [15] and Helmholtz-type [16] equations have been solved using the PWM, coupled with the TSVD regularization technique. An efficient matrix decomposition algorithm for solving the PWM linear system was presented in [17], for axisymmetric Helmholtz problems. More recently, a modification of the PWM basis, spanning the same approximation space but reducing significantly the ill-conditioning of the corresponding linear system was proposed in [18].

In this paper we will consider the numerical solution of the Navier equations of elastodynamics via the MFS and analyze its behavior for source points located far away from the domain of interest. In particular, we will show that the solution of the Navier PDE may be represented in terms of the solutions of two Helmholtz equations and consequently, as a linear combination of shear and compressional elastic plane waves. This representation will allow us to extend the PWM to elastic wave propagation problems.

The use of plane waves for the numerical solution of the time-harmonic elastic wave equation is not new. The advantages of such shape functions in comparison with the standard polynomial finite

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element schemes have been recognized more than two decades ago. Several numerical methods such as the ultra weak variational formulation (UWVF) [19], the discontinuous enrichment method [20] or the variational theory of complex rays [21], among others, have been proposed for simulating elastic wave phenomena. In particular, the use of oscillatory basis functions allows for the accurate solution of medium and high frequency wave propagation problems with lower computational effort, in comparison with the classical FEM. From a theoretical point of view, algebraic order of convergence for linear combinations of elastic plane waves with respect to the dimension of the approximation space and the diameter of the domain has been proven in [22].

The novelty in our work consists in the development and application of a meshfree and integration free method, with elastic plane waves as shape functions, for the numerical solution of the homogeneous Navier equation. The PWM is a Trefftz method and therefore its algorithm is based on the solution of a single collocation linear system for the boundary conditions. Here, the shape functions are operator geared and therefore no numerical or analytical differentiation is also required for the application of the method, as for example if general purpose RBF functions were to be used. In smooth settings, the convergence of the PWM is exponential and thus superior to polynomial finite element methods.

The rest of the paper is organized as follows. In Section 2 we include a brief review of the existing theoretical results for the asymptotic behavior of the classical MFS, when applied to acoustic wave propagation problems. The formulation of the PWM for the numerical solution of the Helmholtz PDE, as introduced in [10], is also recalled. In Section 3 we analyze the asymptotic behavior of the MFS and generalize the PWM from the acoustic (scalar) case to the elastic (vector) case. Section 4 is dedicated to numerical tests illustrating the relation between the asymptotic MFS and the PWM for several elastic wave propagation problems. The convergence and stability of the PWM is also analyzed.

2. Acoustic case

The propagation of a sound wave of small amplitude in a homogeneous isotropic medium $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\Gamma = \partial\Omega$ is modeled by the wave equation. If one is interested in time-harmonic solutions, the problem may be reduced to the solution of a BVP for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \Gamma \end{cases}, \tag{1}$$

where g is the prescribed Dirichlet boundary condition. If $-k^2$ is not an eigenvalue for the Laplace operator in Ω, or equivalently, if k is not a resonance frequency for the BVP in Ω, problem (1) is well posed and, for C^∞ boundary data, it has a unique solution $u \in C^\infty(\Omega)$, e.g. [23,24].

The fundamental solution for the Helmholtz operator is given by

$$\Phi_k(x) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(k|x|), & d = 2 \\[2ex] \dfrac{e^{ik|x|}}{4\pi|x|}, & d = 3 \end{cases} \tag{2}$$

where $H_0^{(1)}$ is the Hänkel function of the first kind and order zero, $|x|$ represents the Euclidian norm in $\mathbb{R}^d$ and i is the imaginary unit. Note that Φ_k exhibits radial symmetry and oscillatory behavior and its real part is singular at $x = 0$.

By shifting the singularity of $\Phi_k(x)$ to an exterior point $y \in \mathbb{R}^d \backslash \bar{\Omega}$, also called a *source point* or *source*, we may define a particular solution for the Helmholtz PDE, given by $\Phi_k(x - y)$, $x \in \Omega$. The classical MFS, e.g. [4,7], consists in approximating the unknown solution of (1) by a linear combination of such fundamental solutions

$$u \approx \tilde{u}(x) = \sum_{j=1}^{n} \alpha_j \Phi_k(x - y_j) \tag{3}$$

where the n source points $y_1, \dots, y_n$ belong to an admissible source point set, also called *artificial boundary* [6], which we will denote by $\hat{\Gamma} \subset \mathbb{R}^d \backslash \bar{\Omega}$. The boundary conditions are then fitted by solving (exactly or approximately) a collocation linear system for a finite set of boundary knots.

2.1. Asymptotic expansions

The following analytic result characterizes the asymptotic behavior of Φ_k for a source point y located far away from the domain of interest Ω, e.g. [25].

Theorem 1. *For $y \in \mathbb{R}^d$ $(d = 2, 3)$ with $|y| \to \infty$ and $x \in \Gamma$ we have*

$$\Phi_k(x - y) = \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{8k\pi}} \dfrac{e^{ik|y|}}{\sqrt{|y|}} \left(e^{-ikx\cdot\hat{y}} + O\left(\dfrac{1}{|y|}\right)\right), & d = 2, \\[3ex] \dfrac{e^{ik|y|}}{4\pi|y|} \left(e^{-ikx\cdot\hat{y}} + O\left(\dfrac{1}{|y|}\right)\right), & d = 3, \end{cases} \tag{4}$$

where[1] *$\hat{y} = y/|y| \in S^{d-1}$ and $(x \cdot y)$ denotes the scalar product in $\mathbb{R}^d$.*

From (4), for $|y| = R >> |\Omega|$ and $x \in \bar{\Omega}$ the fundamental solution $\Phi_k(x - y)$ is asymptotically equivalent to an *acoustic plane wave*

$$\Phi_k(x - y) \sim C_R e^{ikx\cdot\boldsymbol{d}} \tag{5}$$

with the same frequency k, direction of propagation $\boldsymbol{d} = -\hat{y}$ and constant amplitude $C_R = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \frac{e^{ikR}}{\sqrt{R}}$, for $d = 2$ or $C_R = \frac{e^{ikR}}{4\pi R}$, for $d = 3$. Note that C_R is just a scaling factor for the plane wave.

In view of the above result and noting that acoustic plane waves of the form

$$W_k(x, \boldsymbol{d}) := e^{ikx\cdot\boldsymbol{d}}, \quad \boldsymbol{d} \in S^{d-1} \tag{6}$$

represent particular solutions of the Helmholtz PDE (with frequency k) we may conclude that the MFS with source points located far away from the domain of interest is asymptotically equivalent to a Trefftz method based on superposition of acoustic plane waves with unitary directions of propagation. This meshfree method was formulated and analyzed by Alves and Valtchev in [10] and we refer to it as the Plane Waves Method (PWM).

In the Plane Waves Method, the unknown solution of the BVP is approximated by a linear combination of acoustic plane waves

$$u \approx \tilde{u}(x) = \sum_{j=1}^{n} \alpha_j W_k(x, \boldsymbol{d}_j), \quad x \in \bar{\Omega}, \tag{7}$$

where

$$\mathcal{D} = \{\boldsymbol{d}_j \in S^{d-1} \; : \; j = 1, \dots, n\} \tag{8}$$

is a prescribed set of n distinct unitary directions. Noting that the plane waves W_k, and therefore $\tilde{u}$, satisfy the Helmholtz PDE, the unknown coefficients $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ are calculated by enforcing the boundary conditions on a set of boundary collocation points. The resulting collocation linear system is usually ill-conditioned and a pseudo-inversion technique (e.g. TSVD) may be requires for its solution, see [26].

The applicability of the PWM may be justified in terms of density results for linear combinations of plane waves in an appropriate functional space defined on Γ, see [10,25]. From a numerical point of view, the performance of the method was analyzed in [10,11] and its equivalence to the asymptotic case of the MFS has been confirmed. Also, exponential convergence of the PWM with respect to the number of collocation points and unitary directions has been observed for BVP posed in smooth domains and with analytic boundary conditions.

[1] Notation: $B_R^d = \{x \in \mathbb{R}^d \; : \; |x| < R\}$, $S_R^{d-1} = \{x \in \mathbb{R}^d \; : \; |x| = R\}$, $B^d := B_1^d$ and $S^{d-1} := S_1^{d-1}$.