Engineering Analysis with Boundary Elements 000 (2017) 1–11

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Collocation Boundary Element Method for the pricing of Geometric Asian Options

A. Aimi, C. Guardasoni*

Department of Mathematical Physical and Computer Sciences, University of Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy

A R T I C L E I N F O

MSC:
91G60
65M38

Keywords:
Boundary Element Method
Fokker–Planck equation
Geometric Asian options
Barrier options
Greeks

A B S T R A C T

The Semi-Analytical method for pricing of Barrier Options (SABO) already applied in the context of European options is here extended to the evaluation of geometric Asian options with barriers. The validity of this approximation method, based on the use of collocation Boundary Element Method, is illustrated by numerical examples, where accuracy and stability of the presented approach are analyzed.

1. Introduction

The availability of advanced numerical techniques and faster computer systems are often exploited for a more scientific approach to the problem of pricing financial products.

A new algorithm, the so-called SABO (Semi-Analytical method for pricing of Barrier Options), for the computation of European-style barrier options in the Black-Scholes and Heston models has been recently introduced in [1–3] and anticipated in [4,5].

SABO has resulted to be stable and efficient in the special case of “barrier options” as it is based on Boundary Element Method that perfectly suits differential problems defined in unbounded domains whose data are assigned on a limited boundary. Computations are performed with high accuracy because of the implicit satisfaction of the solution far-field behavior and because of the low discretization costs. Moreover, the method provides a straight hedging computation. The essential requisite, that makes it not as general as other numerical methods, is that, for its application, we need the knowledge, at least in an approximated form, of the transition probability density related to the vanilla option problem.

This paper is aimed at implementing and testing the validity of SABO in the evaluation of continuously sampled geometric Asian options with barrier [6].

Asian options are derivative contracts giving the holder the right to buy an asset for its average price over some prescribed period. Accordingly, their payoff at maturity depends on the average value of an underlying asset over some time interval; therefore we must keep track of more information about the asset price path than simply its present position. The average used in the calculation of the option’s payoff can be defined in different ways: it can be an arithmetic average or a geometric average and the data could be discretely sampled or continuously sampled so that every realized asset price over the given period is used. Almost all Asian options are traded among practitioners with arithmetic average, but this work can be conceived as an intermediate and preparatory step, because the study of geometric case can give some information also about the evaluation of Asian barrier options with arithmetic mean (for which it is a lower bound and that can be used as control variate in Monte Carlo simulations) and because the mathematical foundations in the geometric case are well established and numerically easier to treat.

In presence of a “barrier”, Asian option contracts get into existence or extinguish when the underlying asset reaches a certain barrier value.

With this additional condition w.r.t. plain vanilla contracts, the buyer get a reasonable protection against inconvenient fluctuations of the underlying price and the issuer can attain a better forecasting of the terminal position. In general Asian options, and in particular Asian barrier options, are less expensive than corresponding vanilla options and therefore they are more attractive.

For standard Asian options with geometric mean equipped with floating or fixed strike price, closed formula solutions are available [7], but if the contract involves non standard payoffs or arithmetic mean or barriers, numerical techniques are unavoidable. The pricing is then traditionally based on Monte Carlo methods [7], binomial/trinomial methods [8] or on domain methods, such as Finite Volume Methods [9] and Finite Difference methods [10]. Monte Carlo methods are

* Corresponding author.
E-mail address: chiara.guardasoni@unipr.it (C. Guardasoni).

https://doi.org/10.1016/j.enganabound.2017.10.007
Received 19 July 2017; Received in revised form 28 September 2017; Accepted 1 October 2017
Available online xxx
0955-7997/© 2017 Elsevier Ltd. All rights reserved.

affected by high computational costs and inaccuracy due to their slow convergence; domain methods have some troubles concerning stability: for path-dependent options, but also in the simpler Black-Scholes European option framework, there is the problem of degeneracy of the involved differential operator, pointed out for example in [11] and [12], in fact, for small volatility, the pricing PDE is convection dominated, leading to numerical problems in the form of spurious oscillations. For a quite complete survey and careful analysis of numerical methods available for arithmetic and geometric Asian options without barriers, the interested reader is referred to [13].

Anyway, barrier options are largely exchanged, as they are good products for hedging and investment and they are cheaper than vanilla options, but for Asian options we found in literature only the analysis of [14] which provides rigorous bounds in the arithmetic mean case. In this paper we illustrate how efficient, reliable and quite plain the application of SABO to continuously sampled geometric Asian option with barriers is. For clarity, the description is carried out in the case of call options with an up-and-out barrier and numerical examples concern only the case of fixed strike payoff but the method is very general w.r.t. these features. Unfortunately, the same can not be said referring to the extension to continuously sampled arithmetic Asian option, that, from a theoretical point of view, needs only some slight modifications but, practically, it collides with some numerical difficulties that will be the object of our next investigation.

The paper is structured as follows: in Section 2 there is an overview of the model problem, SABO method is described in Section 3, while in Section 4 there are some hints about performing hedging by SABO. At last in Section 5 two numerical examples related to a geometric Asian call option with fixed strike payoff and up-and-out barrier are presented and discussed.

2. The model problem

A geometric Asian option V is an option depending on the evolution of the stock price S_t (through the duration of the contract, assumed to be [0, T]) and on the geometric average of the stock price over some time interval : $\exp(A_t/t)$, having defined

$$A_t := \int_0^t \log(S_t)dt. \tag{1}$$

If the stochastic process S_t is modeled by the usual geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2}$$

where r denotes the risk free interest rate, σ the volatility and W_t a standard Wiener process, then, A_t is a lognormal stochastic process too.

With the classical hedging arguments applied in the Black-Scholes framework [12], it is possibile to conclude that the Asian option value $V(S, A, t)$ solves the following partial differential equation (PDE):

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \log(S)\frac{\partial V}{\partial A} - rV = 0\\ &\qquad S\in\mathbb{R}^+,\ A\in\mathbb{R},\ t\in[0,T)\end{aligned} \tag{3}$$

Different final boundary conditions (payoffs) define different types of contract, such as:

$$\text{floating strike call}\quad V(S,A,T) = \max\left(S - \exp\left(\frac{A}{T}\right), 0\right) \tag{4}$$

$$\text{floating strike put}\quad V(S,A,T) = \max\left(\exp\left(\frac{A}{T}\right) - S, 0\right) \tag{5}$$

$$\text{fixed strike call}\quad V(S,A,T) = \max\left(\exp\left(\frac{A}{T}\right) - E, 0\right) \tag{6}$$

$$\text{fixed strike put}\quad V(S,A,T) = \max\left(E - \exp\left(\frac{A}{T}\right), 0\right) \tag{7}$$

for $S\in\mathbb{R}^+$, $A\in\mathbb{R}$ and E the strike price. Fixed strike Asian options are less expensive than vanilla options and guarantee that the average exchange rate realized during the year is above some level. Floating strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. However, SABO can treat also other more unusual payoffs.

Explicit boundary conditions are not available in literature. Some boundary conditions are implicitly satisfied by V through its payoff behavior and they are such to assure existence and uniqueness of the Cauchy partial differential problem solution (issue that is discussed in Appendix A.1).

Anyway, by stochastic considerations, it is possible to define the exact solution in an integral form as payoff expected value that can be therefore employed also with payoff contracts more general than (4)–(7):

$$V(S,A,t) = \int_{-\infty}^{+\infty}\int_0^{+\infty} V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A}. \tag{8}$$

The function $G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})$ is the *transition probability density function* (PDF), also known as *Green's function* or *fundamental solution* of the partial differential problem: as a function of $(S,A,t)\in\mathbb{R}^+\times\mathbb{R}\times[0,T)$ the PDF solves (3) and, as a function of $(\widetilde{S},\widetilde{A},\widetilde{t})$, it solves the *backward Kolmogorov equation* adjoint of (3): *for each* $(S,A,t)\in\mathbb{R}^+\times\mathbb{R}\times[0,T)$

$$\begin{cases} -\dfrac{\partial G}{\partial\widetilde{t}} + \dfrac{\sigma^2}{2}\widetilde{S}^2\dfrac{\partial^2 G}{\partial\widetilde{S}^2} + (2\sigma^2 - r)\widetilde{S}\dfrac{\partial G}{\partial\widetilde{S}} - \log\left(\widetilde{S}\right)\dfrac{\partial G}{\partial\widetilde{A}} + (\sigma^2 - 2r)G = 0\\ \quad \widetilde{S}\in\mathbb{R}^+,\ \widetilde{A}\in\mathbb{R},\ \widetilde{t} > t\\ G(S,A,t;\widetilde{S},\widetilde{A},t) = \delta\left(S - \widetilde{S}\right)\delta(A - \widetilde{A})\\ \quad \widetilde{S}\in\mathbb{R}^+,\ \widetilde{A}\in\mathbb{R}\end{cases} \tag{9}$$

where $\delta(\cdot,\cdot)$ represents the Dirac distribution[1]. The solution of problem (9) must satisfy suitable boundary conditions assuring that the Green identity[2] is verified. Look at [15] for the Differential Analysis on the matter.

Denoting by $H[\cdot]$ the Heaviside step function, the closed form solution of problem (9) is

$$\begin{aligned} G\left(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}\right) = \frac{\sqrt{3}H\left[\widetilde{t}-t\right]}{\pi\sigma^2\left(\widetilde{t}-t\right)^2}\exp\Bigg\{&-\frac{2}{\sigma^2\left(\widetilde{t}-t\right)}\log^2\left(\frac{S}{\widetilde{S}}\right)\\ &+\frac{6}{\sigma^2(\widetilde{t}-t)^2}\log\left(\frac{S}{\widetilde{S}}\right)\left(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\right)\\ &-\frac{6}{\sigma^2\left(\widetilde{t}-t\right)^3}\left(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\right)^2\\ &-\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-t)\Bigg\}\left(\frac{\widetilde{S}}{S}\right)^{\frac{2r-\sigma^2}{2\sigma^2}}\frac{1}{\widetilde{S}} \end{aligned} \tag{10}$$

that satisfies

$$\int_{-\infty}^{+\infty}\int_0^{+\infty} G\left(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}\right)d\widetilde{S}\,d\widetilde{A} = \exp\left(-r(\widetilde{t}-t)\right). \tag{11}$$

The attainment of expression (10) is related to theoretical results in Appendix A.1.

[1] The *Dirac's delta distribution* satisfies the property that $\int_{-\infty}^{+\infty}\delta(y,x)f(x)dx = f(y),\quad \forall f\in C_0^\infty(\mathbb{R})$.

[2] When considering the PDE $\mathcal{P}[u] = 0$ defined by the partial differential operator $\mathcal{P}$ applied to the unknown solution u then, a function G, satisfies the Green identity if

$$\langle\mathcal{P}[u], G\rangle - \langle u, \mathcal{P}^*[G]\rangle = 0$$

where $\mathcal{P}^*$ is the adjoint of operator $\mathcal{P}$ and $\langle\cdot\rangle$ is the L^2 scalar product.