Engineering Analysis with Boundary Elements 000 (2018) 1–7

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Local multilevel scattered data interpolation

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ARTICLE INFO

MSC:
46E35
65D05
65J05

Keywords:
Scattered data
Radial basis functions
Multilevel interpolation
Sobolev spaces
Native spaces

ABSTRACT

Radial basis functions play an increasingly prominent role in modern approximation. They are widely used in scattered data fitting, numerical solution of partial differential equations, machine learning and others. Although radial basis functions have excellent approximation properties, they often produce highly ill-conditioned discrete algebraic system and lead to a high computational cost. The paper introduces local multilevel scattered data interpolation method, which employ nested scattered data sets and scaled compactly supported radial basis functions with varying support radii. We will provide convergence theory for Sobolev target functions. And several numerical experiments will be provided to conform the efficiency of new method.

1. Introduction

In this work, we introduce and analyze the local multilevel scattered data interpolation method. As an important tool in modern approximation, radial basis functions allow the easy construction of approximation spaces in arbitrary dimensions with arbitrary smoothness. Error estimates for scattered data interpolation via radial basis functions have been proved not only in native spaces, but also in Sobolev spaces. We can see the theory in early papers [11,18] and recent papers [12,13]. Particularly in [14], Schaback compared all linear PDE solvers and found that error-optimal methods are radial basis functions methods. Although globally supported radial basis functions have excellent approximation properties, they often produce dense discrete system which tends to be poor conditioning. Compactly supported radial basis functions leads to a very well-conditioned sparse system, but at the cost of a poor approximation accuracy. This is a "trade-off" principle. That is to say, small support leads to a well-conditioned system but also poor accuracy, while large support yields excellent accuracy at the price of ill-conditioned system.

The goal of this paper is to design a local multilevel scattered data interpolation method which is likely to maintain approximation accuracy and with a low computational cost. To avoid writing constants repeatedly, we shall use notations $\lesssim$, $\gtrsim$ and $\cong$, as in the well-known paper [19]. With some constants c and C, the short notation $x \lesssim y$ means $x \le Cy$; and $x \cong y$ means $cx \le y \le Cx$.

2. Local multilevel interpolation method

As we saw in introduction, there is a "trade-off" principle for interpolation with compactly supported radial basis functions. In order to combine the advantages of globally supported radial basis functions and compactly supported radial basis functions, a multilevel stationary interpolation algorithm was implemented first in [8], and then studied by a number of other researchers [3–5,7,9,10,16]. In multilevel algorithm, the target function first is interpolated on the coarsest level by one of the compactly supported radial basis functions with a larger support. Then the residual can be formed, and be interpolated on the next finer level by the same compactly supported radial basis function but with a smaller support. This process can be repeated and be stopped on the finest level. And the final approximation is the sum of all of interpolants.

The weakness of the multilevel interpolation algorithm is that a global interpolation problem must be solved on each level, which will waste a lot of computational time. To avoid solving a series of global problems, we will use the local multilevel interpolation method.

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ be a finite point set in Ω. We associate this point set with fill distance

$$h_{X,\Omega} := \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_j\in X} \|\mathbf{x} - \mathbf{x}_j\|_2, \tag{2.1}$$

and separation distance

$$q_X := \min_{j\neq k} \|\mathbf{x}_k - \mathbf{x}_j\|_2. \tag{2.2}$$

We assume that the point set is quasi-uniform, which means $h_{X,\Omega} \cong q_X$.

To construct a local method, we need a successive refinement point sets $X_1, X_2, \dots$, which have fill distances $h_j = h_{X_j,\Omega}$. Of course $X_1 \subset X_2 \subset \cdots$, if h_j are monotonically decreasing. Let $X_j^* \subseteq X_j$ be newly added point set in X_j, and have fill distances $h_j^* \ge h_j$. Then we have $X_1 = X_1^*$ and $X_j = X_{j-1} \bigcup X_j^*$ for $j = 2, \dots$.

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https://doi.org/10.1016/j.enganabound.2017.11.017
Received 30 June 2017; Received in revised form 14 September 2017; Accepted 30 November 2017
Available online xxx
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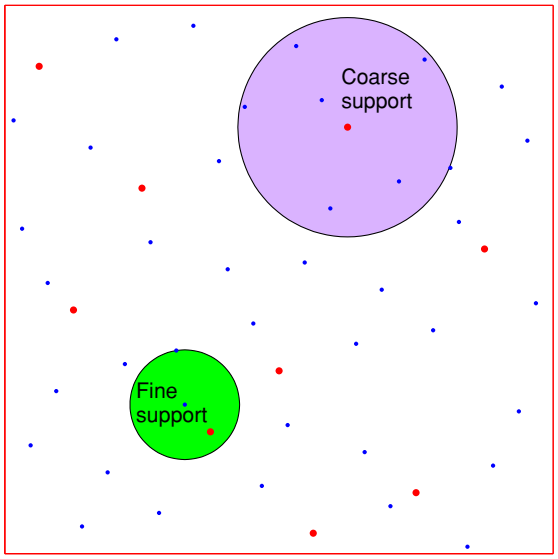

Fig. 1. Nested scattered data sets and corresponding supports.

Table 1
Number of points on each level and corresponding h_j^*.

Level j	1	2	3	4	5	6
$\#(X_j)$	9	34	115	404	1493	5718
$\#(X_j^*)$	9	25	81	289	1089	4225
$h_j^* \approx$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

Table 2
Numerical result for $b = 1$.

j	RMS-error	Max-error	%nonzero	Time (s)
1	2.4843E−01	8.4283E−01	55.56	0.03
2	1.5539E−01	7.9190E−01	15.20	0.05
3	1.0283E−01	7.8592E−01	4.89	0.06
4	8.2009E−02	7.8592E−01	1.10	0.15
5	6.6870E−02	7.6557E−01	0.32	0.84
6	4.5191E−02	7.1481E−01	0.08	8.98

Table 3
Numerical result for $b = 0.5$.

j	RMS-error	Max-error	%nonzero	Time (s)
1	1.3259E−01	4.6615E−01	97.53	0.02
2	4.8543E−02	3.8572E−01	52.32	0.04
3	2.8578E−02	3.5321E−01	15.71	0.08
4	2.2981E−02	3.4422E−01	4.45	0.18
5	1.5945E−02	3.1050E−01	1.19	0.91
6	9.8463E−03	1.9932E−01	0.31	9.10

We pick a kernel $\Phi_j : \Omega \times \Omega \to \mathbb{R}$ for each X_j^*. In many applications, the kernel is given by one of the scaled version of a translation invariant functions. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be one of the compactly supported radial basis functions. We can define the kernel as

$$\Phi_j(\cdot, \mathbf{y}) = \varepsilon_j^d \Phi(\varepsilon_j(\cdot - \mathbf{y})), \quad \mathbf{y} \in X_j^*. \tag{2.3}$$

Clearly, Φ_j is a scaled radial basis function whose support is a ball with radius $\frac{1}{\varepsilon_j}$ and center $\mathbf{y}$ (See Fig. 1).

Then we can build local approximation spaces

$$W_j = \text{span}\{\Phi_j(\cdot, \mathbf{y}) : \mathbf{y} \in X_j^*\}. \tag{2.4}$$

Let V_j be approximation spaces on data sets X_j. Then we have $V_1 = W_1$, $V_j = V_{j-1} \bigoplus W_j$ for $j = 2, \ldots$, and $V_1 \subset V_2 \subset \cdots$.

We need to associate W_j with some kinds of norms. The associated reproducing kernel Hilbert space (or native space) $\mathcal{N}_K(\mathbb{R}^d)$ of kernel K

Table 4
Numerical result for $b = 0.1$.

j	RMS-error	Max-error	%nonzero	Time (s)
1	9.7623E−02	3.1969E−01	100.00	0.03
2	4.6522E−02	2.0613E−01	100.00	0.05
3	7.8327E−03	3.9937E−02	99.94	0.07
4	1.4573E−03	2.3140E−02	65.17	0.20
5	3.7999E−04	7.2034E−03	23.09	1.00
6	1.0553E−04	1.5967E−03	6.67	9.93

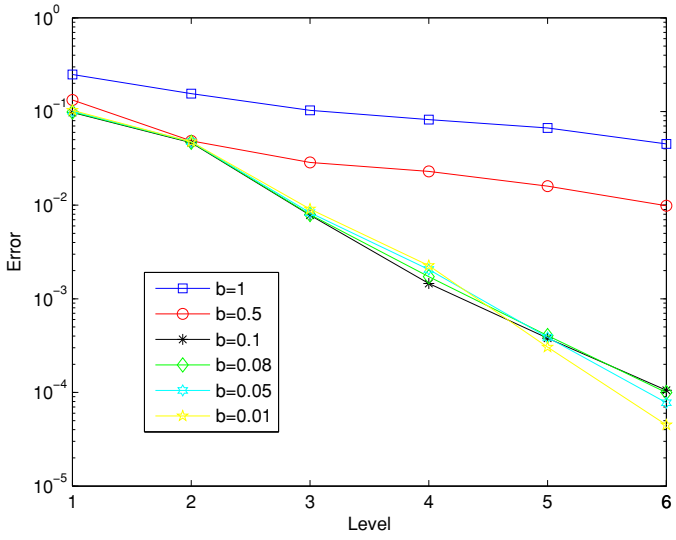

Fig. 2. RMS-error for C^2 local interpolation with varying level and b.

consists of all functions $g \in L^2(\mathbb{R}^d)$ satisfying

$$\|g\|_K^2 = \int_{\mathbb{R}^d} \frac{|\widehat{g}(\boldsymbol{\omega})|^2}{\widehat{K}(\boldsymbol{\omega})} d\boldsymbol{\omega} < \infty \tag{2.5}$$

with

$$\widehat{g}(\boldsymbol{\omega}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} g(\mathbf{x}) e^{-i\mathbf{x}^T\boldsymbol{\omega}} d\mathbf{x}. \tag{2.6}$$

Suppose further that kernel K satisfies

$$\widehat{K}(\boldsymbol{\omega}) \cong (1 + \|\boldsymbol{\omega}\|_2^2)^{-\tau}, \quad \tau > \frac{d}{2} \quad \text{and} \quad \boldsymbol{\omega} \in \mathbb{R}^d \tag{2.7}$$

then $\mathcal{N}_K(\mathbb{R}^d)$ is a Sobolev space $H^\tau(\mathbb{R}^d)$ and it has an equivalent Sobolev type norm

$$\|g\|_{H^\tau(\mathbb{R}^d)}{}^2 = \int_{\mathbb{R}^d} |\widehat{g}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\tau d\boldsymbol{\omega}. \tag{2.8}$$

Let $K = \Phi_j$, then the norm of the space W_j will be denoted by

$$\|g\|_{\Phi_j}^2 = \int_{\mathbb{R}^d} \frac{|\widehat{g}(\boldsymbol{\omega})|^2}{\widehat{\Phi}_j(\boldsymbol{\omega})} d\boldsymbol{\omega}, \quad \forall g \in W_j. \tag{2.9}$$

It is well-known that the Matérn functions and Wendland's compactly supported functions can satisfy (2.7) (or their Fourier transforms decay only algebraically). We can refer to monographs [2,6,15] for the details.

Using the techniques of Fourier transform and the definition of norm ((2.8) and (2.9)), we have following norm equivalence theorem.

Theorem 2.1. *Let Φ be a kernel satisfying (2.7) and Φ_j be defined by (2.3). If $g \in H^\tau(\mathbb{R}^d)$ and every scaling parameter $\varepsilon_j \geq 1$, then $\mathcal{N}_{\Phi_j}(\mathbb{R}^d) = H^\tau(\mathbb{R}^d)$ and*

$$\|g\|_{\Phi_j} \lesssim \|g\|_{H^\tau(\mathbb{R}^d)} \lesssim \varepsilon_j^\tau \|g\|_{\Phi_j}.$$

Proof. See [16], a similar norm equivalence theorem with inverse of ε_j as scaling parameter. □