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Improved localized radial basis functions with fitting factor for dominated convection-diffusion differential equations

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ABSTRACT

Ill-conditioning problem of Global Radial Basis Functions (GRBFs) is a fundamental limitation for approximation of differential equation using this method. Most recently, some researchers have applied Local Radial Basis Functions (LRBFs) to approximate singularly perturbed convention diffusion problems. In these kinds of problems, mostly appear boundary and (or) interior layers. Existence of these regions causes instability and oscillation solutions in numerical methods. To avoid these problems, upwind LRBFs are used. In other words, convective part and diffusion part of equations discretize through upwind LRBFs and central LRBFs method, respectively. Although this technique stabilizes the scheme but reduces the accuracy, by the way, the selection of upwind direction is more complicated. To overcome these problems, in this paper, we multiply an artificial diffusion to diffusion part and apply central scheme to discretize convective part. Since central method for these kinds of problems are unstable, adding artificial diffusion causes the central method is stabilized. This strategy significantly improve the accuracy. For verification, several numerical examples are considered and compared with other schemes. The comparisons show the superiority of the method to tradition approaches. Results of this study reveal that the proposed method can be successfully applied for singularly perturbed differential equations.

1. Introduction

In modeling physical convection-diffusion phenomena, the following typical form of partial differential equations (PDEs) are usually applied:

$$\begin{cases} Lu := -\varepsilon\Delta u(x,y) + b(x,y).\nabla u(x,y) + c(x,y)u(x,y) = f(x,y), & in\ \Omega \subset \mathrm{R}^2, \\ Bu = g(x,y) & on\ \Gamma, \end{cases} \tag{1}$$

where $0 \ll \varepsilon < 1$. Assume $b(x,y) = (b_1(x,y), b_2(x,y)) > (\beta_1, \beta_2) > (0,0)$ and $c(x, y) \geq 0$ on $\bar{\Omega}$ which is closure of domain. We suppose b, c, f are sufficiently smooth functions in Ω. The solution u of (1) has exponential boundary layer at outflow boundaries Γ^+[1]. Boundary layers cause some instability in numerical methods. Additionally, presence of a very small parameter ε in (1) causes a large number of data points and correspondingly computational works and costs are needed to obtain accurate solution. Furthermore, when ε is selected scientifically very small, for instance $\varepsilon = 10^{-6}$ or lower value, we require at least $O\left(\frac{1}{\varepsilon^2}\right)$ points, which may not be practical and affordable.

Researchers have applied some strategies for discretization to avoid using too many data points. In finite difference method (FDM), miller et al. [2] developed a successful upwind finite difference method. Artificial diffusion or fitting factor is also applied to achieve accurate solution and uniformly convergence in ε [3]. Shishkin introduced a point-wise equidistance grid to improve the accuracy of solution [4,5]. For RBFs and PS method trummer [6,7] presented coordinate stretching technique to both improve the accuracy and use less collocation points. A good review on the numerical methods for singularly perturbed differential equation can be found in Roos's book and references in it [1].

Radial basis functions became popular as a very capable technique for interpolation of multidimensional scattered data points [8,9] and approximation of partial differential equations (PDEs) on irregular domains [10,11]. The main advantages of the method are: no need to mesh generation, ease of programing and spectral accuracy, but its main drawback is ill-conditioning of resulting linear system. To overcome this drawback a local version of the method was later proposed. The idea of the LRBFs method is to ignore the spectral accuracy of Global RBFs, in order to have a sparse better-conditioned linear system. This technique seems to be firstly proposed by Tolstykh in 2000 [12]. Some applications of LRBFs were recommended by Liu et al. [13–17]. Analysis of the free vibration for 2D solids had been done and applied to simulate incompressible flow [13,17]. Shu et al. [18] used LRBFs to solve 2D incompressible Navier-Stokes equations. In these papers, to overcome the ill-conditioning of the interpolation matrix, small set of nodes in the

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neighborhood of any collocation node were used to approximate derivatives. In the local support domain, the RBFs are more suitable than polynomial ones. If an inappropriate polynomial basis is chosen for a given set of nodes, it may result in singular moment matrix [19].

Advantages of the local version of the method is its suitability for problems with discontinuous boundary conditions and the main strength of LRBFs method is for irregular shape and complex geometry.

To discretize dominated convection-diffusion, Shan [20] presented upwind LRBFs scheme such as FD method. He illustrated 1D-singularly perturbed differential equations and compared with Global RBFs for small values of ε. Sanyasiraju [21,22] proposed a multiquadric upwind LRBF scheme for convection dominated problems and used a local optimization algorithm to compute the optimal variable shape parameter. Hon [23] combined LRBF collocation method with the partial upwind scheme and applied it to approximate numerical solutions to one and two dimensional Burger's equation with shock waves and singular perturbation problems with turning points and boundary layers. Zahab et al. [24] simulated a laminar incompressible flows applying an LRBFs method with upwind scheme. After that, Shu et al. [25] employed this method to investigate inviscid compressible flows with shock waves. From the numerical results given in [20] it was found that when upwind scheme is applied for singularly perturbed problems, upwind scheme does not work well at nodes near the boundary layer when the perturbation parameter is not small.

The main aim in this paper is to improve the accuracy and maintain the advantages of Radial basis functions such as no need the meshless generation, stability and non-oscillatory numerical solution. To gain this aim, we utilize an artificial diffusion which introduced by I'len [3] and are applied for FDM. Since central method for these kinds of problems are unstable, adding artificial diffusion causes the central method is stablized. We multiply this artificial diffusion to diffusion part of (1) and discretize diffusion part and convection part through central scheme instead of upwind scheme. Numerical results and comparison with exact solution and upwind LRBF methods validate the scheme and indicate that this strategy gives higher accuracy, stable and non-physical oscillation.Therefore, this methodology can be proposed as a better method to solve general stiff problems.

Rest of the paper is organized as follows. In Section 2, we describe the LRBFs formulas and how the unknown weighting coefficients can be determined. Collocation nodes and fitting factors will be investigated in Section 3. In Section 4, LRBFs method and artificial diffusion for boundary layer problems are described. Numerical study and numerically comparison the scheme with other methods are carried out in section 5. Conclusion and summary of this work is given in the final Section.

2. Localized RBFs method

The LRBFs method is considered as a generalized difference scheme to scattered data points. In classical difference schemes, derivatives of a function u at a given point are approximated as linear combinations of the values of u at some surrounding nodes. The derivative $D^m u(\mathbf{x})$ with respect to $\mathbf{x} = (x, y)$ at a point $\mathbf{x}_i$ is approximated by

$$D^m u(\mathbf{x}_i) \approx \sum_{j=1}^{n_j} w_{ij}^m u(\mathbf{x}_{i_j}), \tag{2}$$

where $w_{ij}^{(m)}$ represents the weight coefficients, n_j is the number of points in the local domain and $D^m = \frac{\partial^m}{\partial^{m_1} x_1 \partial^{m_2} x_2 \ldots \partial^{m_l} x_l}$ where $m_1 + m_2 + \cdots + m_l = m$. The unknown weight coefficients $w_{ij}^{(m)}$ are computed using RBFs interpolation [26]. Using RBFs, the relation (2) changes to the following form,

$$D^m \varphi_k(\mathbf{x}_i) = \sum_{j=1}^{n_j} w_{ij}^m \varphi_k(\mathbf{x}_{i_j}), \quad k = i_1, i_2, \ldots, i_{n_j} \tag{3}$$

Table 1
The definitions of some radial basis functions.

Name of functions	Definition
Multiquadric (MQ)	$\varphi(r,c) = \sqrt{1+c^2r^2}$
Inverse multiquadric(IMQ)	$\varphi(r,c) = \frac{1}{\sqrt{1+c^2r^2}}$,
Inverse quadratic (IQ)	$\varphi(r,c) = \frac{1}{1+c_2^2r^2}$
Gaussian (GA)	$\varphi(r,c) = e^{-c^2r^2}$.

The function φ_k in (3) is the Radial Basis Functions. The common choices for RBFs are listed in Table 1. In addition, we utilize Multiquadrics Radial Basis Function (MQRBF),

$$\varphi(r,c) = \sqrt{1+c^2r^2},$$

where $r^2 = ||\mathbf{x} - \mathbf{x}_i||_2^2$, c is the shape parameter and $||.||_2$ denotes the usual Euclidean norm between nodes $\mathbf{x}$ and $\mathbf{x}_i$ The unknown weight coefficients $w_{ij}^{(m)}$ can be approximated by solving the following linear system

$$\psi W = D^m \psi \tag{4}$$

where $D^m\psi = [D^m\varphi_{i_1}(\mathbf{x}_i), D^m\varphi_{i_2}(\mathbf{x}_i), \ldots, D^m\varphi_{i_{n_j}}(\mathbf{x}_i)]^T$, $W = [w_{i1}^{(m)}, w_{i2}^{(m)}, \ldots, w_{in_j}^{(m)}]^T$ and

$$\psi = \begin{bmatrix} \varphi_{i_1}(\mathbf{x}_{i_1}) & \varphi_{i_1}(\mathbf{x}_{i_2}) & \ldots & \varphi_{i_1}(\mathbf{x}_{i_{n_j}}) \\ \varphi_{i_2}(\mathbf{x}_{i_1}) & \varphi_{i_2}(\mathbf{x}_{i_2}) & \ldots & \varphi_{i_2}(\mathbf{x}_{i_{n_j}}) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \varphi_{i_{n_j}}(\mathbf{x}_{i_1}) & \varphi_{i_{n_j}}(\mathbf{x}_{i_2}) & \ldots & \varphi_{i_{n_j}}(\mathbf{x}_{i_{n_j}}) \end{bmatrix} \tag{5}$$

The unknown weights coefficient $w_{ij}^{(m)}$ can be expressed in terms of function values at these local interpolation nodes, i.e.,

$$W = \psi^{-1} D^m \psi \tag{6}$$

Hence the LRBF approximation (2) at $\mathbf{x}_i$ can be rewritten as

$$D^m u(\mathbf{x}_i) \approx D^m \hat{u}(\mathbf{x}_i) = \psi^{-1} D^m \psi u^i, \tag{7}$$

where $u^i = [u(\mathbf{x}_{i_1}), u(\mathbf{x}_{i_2}), \ldots, u(\mathbf{x}_{i_{n_j}})]^T$.

3. Collocation nodes and fitting factors

In the Local Radial Basis Functions, to choose nodes in the influence domain, some considerations such as: the number, symmetry and location of nodes should be taken into account [27]. To approximate the spatial derivatives, two schemes are commonly used, central and upwind schemes. In central schemes the collocation node lies in the center of the influence domain and interpolation nodes are placed symmetrically aside. In Fig. 1 the four most commonly used local techniques are displayed.

In upwind schemes, we move in the upwind direction of the collocation nodes which direction is determined by the sign of the coefficient of convection term (β).

As an example, it can be seen in Fig. 2, in one-dimensional space, there are two directions of each collocation nodes (left and right) in influence domain. All interpolation nodes are chosen on either left or right side of each collocation point.

In two-dimensional space, the selection of upwind scheme becomes more complicated. In finite difference methods, Kopteva [28] presented a 2-noded upwind, which is shown in Fig. 3. Tabata [29] firstly introduced an upwind scheme in finite element method. Some techniques and their details for upwind schemes in two-dimensional can be found in [30]. In Fig. 4, some of these techniques are displayed.

In this study, central scheme is employed to discretize convection part unlike upwind schemes. Since the central scheme leads to non-physical oscillations in the computed solution and numerical method will be unstable for these kinds of problems, to avoid this problem,