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The numerical solution for the time-fractional inverse problem of diffusion equation

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A R T I C L E   I N F O

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A B S T R A C T

In this study, the spectral meshless radial point interpolation (SMRPI) technique is applied to the Cauchy problem of two-dimensional fractional diffusion equation. We obtain the unknown data on the inner boundary when overspecified boundary data is imposed on the outer boundary. The SMRPI is based on a combination of meshfree methods and spectral collocation techniques. The point interpolation method with the help of radial basis functions is used to construct shape functions which act as basis functions in the frame of SMRPI. Here, similar to other meshless methods, localization in SMRPI can reduce the ill-posedness of the Cauchy problem. However, it does not require to use regularization algorithms and therefore reduces computational time. Two numerical examples, are tested to show that the SMRPI can overcome the ill-posedness of the Cauchy problem and has acceptable accuracy. Also, by adding some large perturbations, the proposed method is still stable.

1. Introduction

Fractional diffusion equations arise in various scientific and engineering problems related to anomalous diffusion (superdiffusion, non-Gaussian diffusion, subdiffusion) [1–4], which might be inconsistent with the classical Brownian motion model. It usually appears in mathematics, physics, chemistry, biological systems and so on. Moreover, time-fractional diffusion equation is often used to describe viscoelastic and viscoplastic flow [5,6]. The inverse problems of fractional diffusion equation have been considered by many researchers in many theoretical papers. For a rather incomplete list, the uniqueness of an inverse problem for a one-dimensional fractional diffusion equation was given in [7]. Cheng and Fu [8] gave an iteration regularization method for a time-fractional inverse diffusion problem. Zheng and Wei [9,10] investigated a time-fractional inverse diffusion problem by using a spectral regularization method and a modified equation technique. Liu and Feng [11] gave a modified kernel method for a time-fractional inverse diffusion problem. Shakeri and Dehghan [12] applied the homotopy perturbation method for solving an inverse parabolic equation and computing an unknown time-dependent parameter. Xiong et al. [13] investigated a fractional inverse heat conduction problem defined in a semi-infinite two-dimensional domain and applied the Fourier transform for stability analysis and error estimate.

Yan and Yang [14] investigated two-dimensional time-fractional inverse diffusion problems by an efficient Kansa-type method of fundamental solutions (MFS-K) for solving the Cauchy problem associated with the inhomogeneous elliptic-type equation. Cauchy problems are typical inverse problems, since no information about the solution are available on a part of the boundary. Due to the missing boundary conditions, the solution of these problems does not depend continuously on the data and so are known as ill-posed problems in the sense that small perturbations in the input data may result in enormous deviations in the solution. The present paper considers the following time-fractional diffusion equation:

$$ {}^{c}u_t^{\alpha}(\mathbf{x},t) = \Delta u(\mathbf{x},t) + f(\mathbf{x},t), \qquad \mathbf{x}\in\Omega\subset\mathbb{R}^2, t\in(0,T], \tag{1}$$

$$ u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{2}$$

$$ u(\mathbf{x},t) = g(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_{out}, t>0, \tag{3}$$

$$ \frac{\partial u(\mathbf{x},t)}{\partial n} = h(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_{out}, t>0, \tag{4}$$

where Ω is a doubly connected domain, $\partial\Omega=\Gamma_{out}\cup\Gamma_{in}$ is the whole boundary for problem, Γ_{in} and Γ_{out} are inner and outer boundaries, respectively. Here ${}^{c}u_t^{\alpha}(\mathbf{x},t)$, $0<\alpha<1$, denotes the Caputo fractional derivative of order α with respect to t and it is defined by

$$ {}^{c}u_t^{\alpha}(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(\mathbf{x},\eta)}{\partial\eta}\frac{d\eta}{(t-\eta)^{\alpha}}, \qquad 0<\alpha<1, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function. Also, Δ denotes the Laplacian operator, n is outward unit normal on Γ_{out}, moreover u_0, f, g and h are the

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given functions. Eqs. (3) and (4) are the Dirichlet boundary condition and the Neumann boundary condition, respectively which are imposed on Γ_{out}, and no boundary condition is given on Γ_{in}. Although this inverse problem may have a unique solution, it is well-known that this solution is unstable against small perturbations on the accessible boundary Γ_{out}, see Hadamard [15]. Therefore, Cauchy problems are ill-posed inverse problems. In practical applications, the given Cauchy data usually contains certain noise. Instead of the exact data g and h, we will imposing a random noise on the Cauchy data as following equations:

$$g^{\epsilon}(\mathbf{x},t) = g(\mathbf{x},t)\Big(1+\sigma\epsilon\Big), \qquad \mathbf{x}\in\Gamma_{out}, t>0, \tag{6}$$

$$h^{\epsilon}(\mathbf{x},t) = h(\mathbf{x},t)\Big(1+\sigma\epsilon\Big), \qquad \mathbf{x}\in\Gamma_{out}, t>0, \tag{7}$$

where ϵ is a random number whose range is $[-1,1]$. Also, σ is a user-defined parameter to denote the percentage of the noise.

The main shortcoming of mesh-based methods such as the finite element method (FEM) [16], the finite volume method (FVM) [17] and the boundary element method (BEM) [18] is that these numerical methods rely on meshes or elements. In the two last decades, in order to overcome the mentioned difficulties some techniques so-called meshless methods have been proposed. A brief review of the meshless method has been studied in [19].

In spite of great benefits in using the meshless weak form methods, there are some limitations. For example, the complicated nature of the non-polynomial shape functions may be computationally expensive to implement in a numerical integration scheme. On the other hand, some methods such as those that are based on moving least squares (MLS) and RBFs, need to determine a shape parameter which plays the important role in the accuracy of the methods. Furthermore, the resultant linear systems might be ill-conditioned and to overcome this defect, some regularization methods are needed. In the meshless method based on strong form, such as Kansa's method, this RBF collocation approach is inherently meshless, easy-to-program, and mathematically very simple to learn, but its fundamental flaw is un-stability because of the use of the global strong form. To overcome these shortages, we propose a new spectral meshless radial point interpolation (SMRPI) method which is based on meshless radial point interpolation and spectral collocation techniques [20–22]. In the SMRPI method, the point interpolation method by the help of radial basis functions is proposed to construct shape functions which have Kronecker delta function property and are used as basis functions in the frame of the SMRPI. Based on the spectral methods, evaluation of high-order derivatives of given differential equation is easy by constructing and using operational matrices. The SMRPI method does not require any kind of integration locally over small quadrature domains nor regularization techniques. Therefore, the computational cost of the SMRPI method is less expensive. The aim of this paper is the development of spectral meshless radial point interpolation to obtain the solution of the Cauchy problem of two-dimensional fractional diffusion equation in some doubly connected domains. When overspecified boundary data are imposed on the outer boundary, we obtain the unknown data on the inner boundary.

The outline of this paper is as follows. In Section 2, we introduce the spectral meshless radial point interpolation scheme briefly so that the high order operational matrices are obtained. A time discrete scheme for implementation of the SMRPI is given in Section 3. In Section 4, we report the numerical experiments of solving Eq. (1) for two test problems. Finally a conclusion is given in Section 5.

2. Proposed method

This section has been adapted from Ref. [23]. Consider a continuous function $u(\mathbf{x})$ defined in a domain Ω, which is represented by a set of field nodes. The $u(\mathbf{x})$ at a point of interest $\mathbf{x}$ is approximated in the form of

$$u(\mathbf{x}) = \sum_{i=1}^{n} R_i(\mathbf{x})a_i + \sum_{j=1}^{m} p_j(\mathbf{x})b_j = \mathbf{R}^{tr}(\mathbf{x})\mathbf{a} + \mathbf{P}^{tr}(\mathbf{x})\mathbf{b}, \tag{8}$$

where $R_i(\mathbf{x})$ is a radial basis function (RBF), n is the number of RBFs, $p_j(\mathbf{x})$ is monomial in the space coordinate $\mathbf{x}$, and m is the number of polynomial basis functions. Coefficients a_i and b_j are unknown which should be determined. In the current work, we use the multi-quadrics (MQ) as radial basis functions in Eq. (8) which is defined as follows:

$$\sqrt{r^2+c^2}, \tag{9}$$

where the term c is known as the 'shape parameter'. In order to determine a_i and b_j in Eq. (8), a support domain is formed for the point of interest at $\mathbf{x}$, and n field nodes are included in the support domain (see Fig. 1) (support domain is usually a disk with radius r_s). Coefficients a_i and b_j in Eq. (8) can be determined by enforcing Eq. (8) to be satisfied at these n nodes surrounding the point of interest $\mathbf{x}$. Therefore, by the idea of interpolation Eq. (8) is converted to the following form:

$$u(\mathbf{x}) = \mathbf{\Phi}^{tr}(\mathbf{x})\mathbf{U}_s = \sum_{i=1}^{n} \phi_i(\mathbf{x})u_i, \tag{10}$$

where $\phi_i(\mathbf{x})$'s are called the RPIM shape functions which have the Kronecker delta function property, that is

$$\phi_i(\mathbf{x}_j) = \begin{cases} 1, & i=j, \quad j=1,2,\dots,n, \\ 0, & i\neq j, \quad i,j=1,2,\dots,n. \end{cases} \tag{11}$$

This is because the RPIM shape functions are created to pass thorough nodal values. Moreover, the shape functions are the partitions of unity, i.e.

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1, \tag{12}$$

for more details about RPIM shape functions and the way they are constructed, the readers are referred to see [24]. Now, we construct operational matrices which are essential tools of present approach. Operational matrices make the technique more appropriate to handle partial differential equations with high derivatives. Suppose that the number of total nodes covering the domain of the problem i.e. $\overline{\Omega} = (\Omega\cup\partial\Omega)$ is N. On the other hand, we know that n is depend on point of interest $\mathbf{x}$ (so, after that we call it $n_{\mathbf{x}}$) in Eq. (10) which is the number of nodes included in support domain $\Omega_{\mathbf{x}}$ corresponding to the point of interest $\mathbf{x}$ (for example $\Omega_{\mathbf{x}}$ can be a disk centered at $\mathbf{x}$ with radius r_s, see Fig. 1). Therefore, we have $n_{\mathbf{x}} \le N$ and Eq. (10) can be modified as

$$u(\mathbf{x}) = \mathbf{\Phi}^{tr}(\mathbf{x})\mathbf{U}_s = \sum_{j=1}^{N} \phi_j(\mathbf{x})u_j. \tag{13}$$

In fact, corresponding to node $\mathbf{x}$ there is a shape function $\phi_j(\mathbf{x})$, $j = 1,2,3,\dots,N$, we define $\Omega^c_{\mathbf{x}} = \{\mathbf{x}_j : \mathbf{x}_j \notin \Omega_{\mathbf{x}}\}$ then it is clear from the previous section that

$$\forall \mathbf{x}_j \in \Omega^c_{\mathbf{x}} : \phi_j(\mathbf{x}) = 0. \tag{14}$$

The derivatives of $u(\mathbf{x})$ are easily obtained as

$$\frac{\partial u(\mathbf{x})}{\partial x} = \sum_{j=1}^{N} \frac{\partial \phi_j(\mathbf{x})}{\partial x}u_j, \qquad \frac{\partial u(\mathbf{x})}{\partial y} = \sum_{j=1}^{N} \frac{\partial \phi_j(\mathbf{x})}{\partial y}u_j, \tag{15}$$

and also high derivatives of $u(\mathbf{x})$ are easily given as

$$\frac{\partial^s u(\mathbf{x})}{\partial x^s} = \sum_{j=1}^{N} \frac{\partial^s \phi_j(\mathbf{x})}{\partial x^s}u_j, \qquad \frac{\partial^s u(\mathbf{x})}{\partial y^s} = \sum_{j=1}^{N} \frac{\partial^s \phi_j(\mathbf{x})}{\partial y^s}u_j, \tag{16}$$

where $\frac{\partial^s(\cdot)}{\partial x^s}$ and $\frac{\partial^s(\cdot)}{\partial y^s}$ are s'th derivative with respect to x and y, respectively. Denoting $u_x^{(s)}(\cdot) = \frac{\partial^s(\cdot)}{\partial x^s}$ and $u_y^{(s)}(\cdot) = \frac{\partial^s(\cdot)}{\partial y^s}$, and setting $\mathbf{x} = \mathbf{x}_i$ in