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A new singular element for evaluating stress intensity factors of V-shaped notches under mixed-mode load

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ABSTRACT

In this paper, a new singular element is presented to evaluate stress intensity factors of V-shaped notches subjected to mixed-mode load. The proposed element takes into account special variation of displacements in the vicinity of the notch tip. The singularity at notch tip is variable unlike the crack problem where the displacements around the crack tip have variation of square root of r. In the proposed method, special basis functions considering the singularity order at notch tip are incorporated into the shape functions of the new element, and the singularity order is determined by the included angle of the notch. With the new element, more accurate displacement and stress fields in the neighborhood of the notch tip can be obtained, thus the stress intensity factors are computed more accurately. Accurate stress intensity factors are important for the V-notched structures to develop a fracture criterion. Numerical examples have demonstrated the accuracy and efficiency of the proposed method.

1. Introduction

Studying of V-notched structures is of great importance, since they are stress raisers. The stress at the tip of a sharp notch is singular according to the linear elastic theory. Thus the stresses evaluated at singular point have little reference value and the classical strength theories are not suitable for V-notch problems. The fracture criterion for the V-notched structures should be based on the stress intensity factors.

The boundary element method (BEM) is an attractive method for the V-notch problems due to accurate results for stresses and mesh reduction [1–15]. Rzasnicki et al. [16] applied BEM to analysis of single-edge notch subjected to pure bending. Portela et al. [17] developed a boundary element singularity subtraction technique to analyze the sharp notched plates. Niu et al. [18] proposed an interpolating matrix method coupled with conventional BEM to model singular stress field in V-notched structures. Cheng et al. [19] analyzed the singularity order of V-notch with angularly inhomogeneous elastic properties. In these methods, complicated mathematical deductions are used and they are not convenient to implement in ordinary BEM programs.

In this paper, a new singular element with special shape functions is proposed for evaluating the stress intensity factors of V-shaped notches. The element with usual shape functions cannot accurately model the displacement field around the notch tip unless extremely fine meshes are used. The singular element for crack problems is also not suitable for analyzing the structure with V-shaped notches. This is because the displacements in the vicinity of the notch tip are of the variation of r^{λ}. r is the distance to the notch tip and λ is the eigenvalue ($0.5 \le \lambda \le 1$). The crack tip singular element [20–23] can only model the variation of $r^{0.5}$.

The stress singularity is mainly determined by the first eigenvalue λ_1, especially for large included angle of the notch. In the proposed method, the variation of $r^{\lambda 1}$ is considered in the new singular element. λ_1 varies with respect to the notch angle. The special shape functions are derived according to the notch angle. With the new singular element, more accurate displacement and stress distributions in the neighborhood of the notch tip can be obtained, thus the stress intensity factor is evaluated more accurately. Accurate stress intensity factor is important for the V-notched structures to develop a fracture criterion.

This paper is organized as follows. In Section 2, the BEM is briefly described. Section 3 introduces the new singular element in detail. Numerical examples are given in Section 4. The paper ends with conclusions in Section 5.

2. Boundary element method

2.1. Boundary integral equation

The boundary integral equation for 2D elastostatic problem in an isotropic, homogeneous medium is as follows:

$$c_{ij}(P)u_j(P) = \int_{\Gamma} u^*_{ij}(P,Q)t_j(Q)d\Gamma(Q) - \int_{\Gamma} t^*_{ij}(P,Q)u_j(Q)d\Gamma(Q) \tag{1}$$

where P and Q are the source and the field points, respectively. $c_{ij}(P)$ is a coefficient matrix depending on the smoothness of the boundary Γ at the

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source point P. u_j and t_j represent the displacement and traction components, respectively. $u^*_{ij}(P,Q)$ and $t^*_{ij}(P,Q)$ are the well-known Kelvin fundamental solutions and given by

$$u^*_{ij}(P,Q) = \frac{1}{8\pi G(1-\upsilon)}\left[(3-4\upsilon)\delta_{ij}\ln\frac{1}{r} + r_{,i}r_{,j}\right] \tag{2}$$

$$t^*_{ij}(P,Q) = -\frac{1}{4\pi(1-\upsilon)r}\left\{\frac{\partial r}{\partial n}\left[(1-2\upsilon)\delta_{ij} + 2r_{,i}r_{,j}\right] - (1-2\upsilon)(r_{,i}n_j - r_{,j}n_i)\right\} \tag{3}$$

where G and v are the shear modulus and the Poisson's ratio, respectively. r is the distance between the source and the field point. n_i and n_j are the components of the normal n.

2.2. Solution of the boundary integral equation

Eq. (1) is discretized by n_e elements. The discretization form of the boundary integral equation is given by

$$c_{ij}(P)u_j(P) = \sum_{e=1}^{n_e}\left\{\sum_{\alpha=1}^{n_\alpha} t_j^\alpha \int_{\Gamma_e} u^*_{ij}(P,Q)N_\alpha(Q)d\Gamma(Q)\right\} - \sum_{e=1}^{n_e}\left\{\sum_{\alpha=1}^{n_\alpha} u_j^\alpha \int_{\Gamma_e} t^*_{ij}(P,Q)N_\alpha(Q)d\Gamma(Q)\right\} \tag{4}$$

where n_α is the number of the element nodes. N_α is the shape function of the αth node of the element.

The system of linear algebraic equations can be expressed in matrix form as

$$\mathbf{Hu} = \mathbf{Gt} \tag{5}$$

where vectors **u** and **t** consist of all nodal displacements and tractions on the boundary. Matrix **H** contains integrals involving t^*_{ij}, and matrix **G** contains integrals involving u^*_{ij}. Rearranging Eq. (5) according to the boundary conditions, the final system of linear equations can be obtained.

$$\mathbf{Ax} = \mathbf{f} \tag{6}$$

where **A** is the coefficient matrix. **x** is the vector containing the boundary unknowns at the source nodes. **f** is the known vector on the right-hand side.

3. New singular element

3.1. Analysis of the singularity for the V-shaped notch

The asymptotic stress fields around the notch tip are given by [24]

$$\sigma_r = \frac{S_{\mathrm{I}}}{\sqrt{2\pi}(r)^{1-\lambda_1}}\left\{-\cos(1+\lambda_1)\theta - \frac{(3-\lambda_1)\sin(1+\lambda_1)\alpha}{(1-\lambda_1)\sin(1-\lambda_1)\alpha}\cos(1-\lambda_1)\theta\right\} + \frac{S_{\mathrm{II}}}{\sqrt{2\pi}(r)^{1-\lambda_2}}\left\{\sin(1+\lambda_2)\theta + \frac{(3-\lambda_2)\sin(1+\lambda_2)\alpha}{(1+\lambda_2)\sin(1-\lambda_2)\alpha}\sin(1-\lambda_2)\theta\right\} \tag{7}$$

$$\sigma_\theta = \frac{S_{\mathrm{I}}}{\sqrt{2\pi}(r)^{1-\lambda_1}}\left\{\cos(1+\lambda_1)\theta - \frac{(1+\lambda_1)\sin(1+\lambda_1)\alpha}{(1-\lambda_1)\sin(1-\lambda_1)\alpha}\cos(1-\lambda_1)\theta\right\} + \frac{S_{\mathrm{II}}}{\sqrt{2\pi}(r)^{1-\lambda_2}}\left\{\sin(1+\lambda_2)\theta - \frac{\sin(1+\lambda_2)\alpha}{\sin(1-\lambda_2)\alpha}\cos(1-\lambda_2)\theta\right\} \tag{8}$$

$$\sigma_{r\theta} = \frac{S_{\mathrm{I}}}{\sqrt{2\pi}(r)^{1-\lambda_1}}\left\{\sin(1+\lambda_1)\theta - \frac{\sin(1+\lambda_1)\alpha}{\sin(1-\lambda_1)\alpha}\sin(1-\lambda_1)\theta\right\} + \frac{S_{\mathrm{II}}}{\sqrt{2\pi}(r)^{1-\lambda_2}}\left\{\cos(1+\lambda_2)\theta - \frac{(1-\lambda_2)\sin(1+\lambda_2)\alpha}{(1+\lambda_2)\sin(1-\lambda_2)\alpha}\cos(1-\lambda_2)\theta\right\} \tag{9}$$

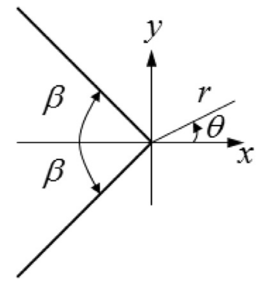

Fig. 1. V-shaped notch.

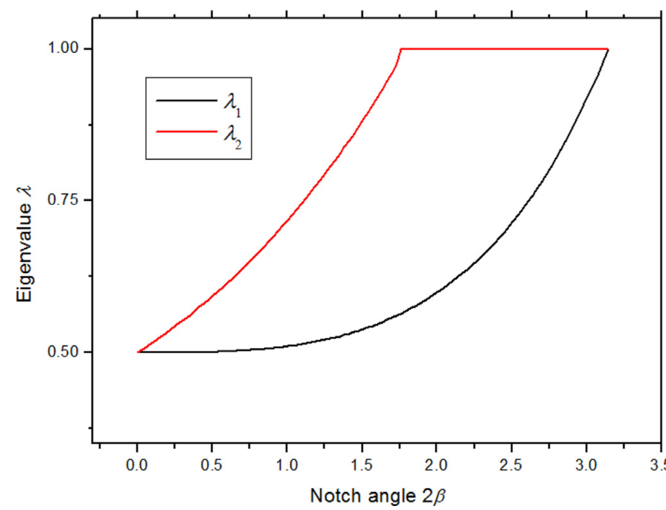

Fig. 2. Variations of λ_1 and λ_2 with respect to the notch angle 2β.

The asymptotic displacement fields are as follows [24]:

$$u_r = \frac{S_I(r)^{\lambda_1}}{\sqrt{2\pi}G}\left\{-\frac{1}{2\lambda_1}\cos(1+\lambda_1)\theta + \frac{\sin(1+\lambda_1)\alpha}{(1-\lambda_1)(1-t)\sin(1-\lambda_1)\alpha}\cos(1-\lambda_1)\theta\right\} + \frac{S_{II}(r)^{\lambda_2}}{\sqrt{2\pi}G}\left\{-\frac{1}{2\lambda_2}\sin(1+\lambda_2)\theta + \frac{\sin(1+\lambda_2)\alpha}{(1+\lambda_2)(1-t)\sin(1-\lambda_2)\alpha}\sin(1-\lambda_2)\theta\right\} \tag{10}$$

$$u_\theta = \frac{S_{\mathrm{I}}(r)^{\lambda_1}}{\sqrt{2\pi}G}\left\{\frac{1}{2\lambda_1}\sin(1+\lambda_1)\theta - \frac{t\sin(1+\lambda_1)\alpha}{(1-\lambda_1)(1-t)\sin(1-\lambda_1)\alpha}\sin(1-\lambda_1)\theta\right\} + \frac{S_{\mathrm{II}}(r)^{\lambda_2}}{\sqrt{2\pi}G}\left\{-\frac{1}{2\lambda_2}\cos(1+\lambda_2)\theta + \frac{t\sin(1+\lambda_2)\alpha}{(1+\lambda_2)(1-t)\sin(1-\lambda_2)\alpha}\cos(1-\lambda_2)\theta\right\} \tag{11}$$

where r, θ denotes a polar co-ordinate system centered at the notch tip as shown in Fig. 1; S_{I}, S_{II} and t are constants; G is the shear modulus; the included angle of the notch is 2β and $\alpha = \pi - \beta$; eigenvalues λ_1 and λ_2 are determined by following characteristic equations:

$$\lambda_1\sin(2\alpha) + \sin(2\lambda_1\alpha) = 0 \tag{12}$$

$$\lambda_2\sin(2\alpha) - \sin(2\lambda_2\alpha) = 0 \tag{13}$$

The stress intensity factors are defined as follows:

$$K_{\mathrm{I}} = \lim_{r\to 0}\sqrt{2\pi}(r)^{1-\lambda_1}\sigma_\theta|_{\theta=0} \tag{14}$$

$$K_{\mathrm{II}} = \lim_{r\to 0}\sqrt{2\pi}(r)^{1-\lambda_2}\sigma_{r\theta}|_{\theta=0} \tag{15}$$

λ_1 and λ_2 vary with respect to the notch angle as shown in Fig. 2. From this figure, we can see that the stress singularity is mainly determined by λ_1, especially for large notch angle. This feature provides convenience for us to get the special shape functions of the new element.