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The generalized finite difference method for an inverse time-dependent source problem associated with three-dimensional heat equation

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ABSTRACT

This paper presents a meshless numerical scheme for recovering the time-dependent heat source in general three-dimensional (3D) heat conduction problems. The problem considered is ill-posed and the determination of the unknown heat source is achieved here by using the boundary condition, initial condition and the extra measured data from a fixed point placed inside the domain. The extra measured data are used to guarantee the uniqueness of the solution. The generalized finite difference method (GFDM), a recently-developed meshless method, is then adopted to solve the resulting time-dependent boundary-value problem. In our computations, the second-order Crank–Nicolson scheme is employed for the temporal discretization and the proposed GFDM for the spatial discretization. Several benchmark test problems with both smooth and piecewise smooth geometries have been studied to verify the accuracy and efficiency of the proposed method. No need to apply any well-known regularization strategy, the accurate and stable solution could be obtained with a comparatively large level of noise.

1. Introduction

The problem of recovering the unknown heat sources arising in time-dependent heat conduction equations presents an interesting challenge in many areas of science and engineering. Specific applications can be found, for example, in aerospace, chemical, mechanical and nuclear engineering [1,2]. The problem belongs to the broad class of inverse source problems which are usually ill-posed because small random errors in measurement may result in arbitrarily large errors in the numerical solutions [3–5]. The existence and uniqueness of solutions for this class of inverse problems have been discussed by Savateev in Ref. [6], when some *priori* information is available on the functional form of the unknown sources.

Some numerical techniques for determining the unknown sources in a parabolic equation have been considered by many authors. In Refs. [7–19], the identification of unknown sources in steady-state heat conduction problems was considered. In Refs. [20–28], several numerical schemes have been proposed to recover the unknown heat sources in transient heat conduction problems in which the heat source is taken to be time-dependent only. The problems of recovering a heat source dependent only on space were considered in Refs. [29–31]. In Ref. [32] the method of fundamental solutions (MFS) coupled with method of radial basis functions (RBFs) has been employed for an inverse heat source problem, without any restriction for the form of unknown sources. In some more recent studies [33,34], inverse source problems for fractional diffusion equations have been considered. Impressive results have been obtained from aforementioned techniques, however only a limited number of papers devoted to 3D transient heat conduction problems are available in the literature. The recovery of heat sources in this subject is one of the purposes of this paper.

In this paper, we investigate a numerical scheme based on the generalized finite difference method (GFDM), a relatively new meshless method, for the recovery of the time-dependent unknown heat source in 3D heat conduction problems. The basis of the GFDM was proposed in the 80s by Lizska and Orkisz [35,36] and were later essentially extended and improved by Benito, Urena and Gavete [37–41]. The main idea of the method is to combine the Taylor series expansions and the moving-least squares (MLS) approximation to derive explicit formulae for the required partial derivatives of unknown variables. In Refs. [37,39], Benito et al. proposed GFDM formulae for second-order partial differential equations in two dimensions. The influence of key parameters, which involved criterions of point generation, weighting function and the shape of the domain, has been well-studied, which can be viewed as a good guidance for using the GFDM. An h-adaptive algorithm for GFDM were described in Refs. [38,42] for 2D and 3D cases, respectively. Ureña et al. [43] studied the GFDM solution for advection-diffusion equations and

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further extended the method to solve third- and fourth-order partial differential equations in Ref. [44]. In 2013, Gavete et al. [45] show the application of the GFDM to dynamic analysis of several problems. In a more recent study, Gavete et al. [41] described how to solve second-order non-linear elliptic partial differential equations using the GFDM. The GFDM was also discussed and extended by many other authors, such as Fan et al. [46] for solving inverse biharmonic boundary-value problems, Chan et al. [47] for 2D non-linear obstacle problems, Gu et al. [10] for inverse steady-state heat conduction problems, and Hua et al. [48] for inverse Helmholtz problems. The goal of this paper is to apply the GFDM to the identification of the unknown sources in 3D time-dependent heat source problems. To our knowledge, this is the first time that the GFDM is extended to solve this kind of inverse problems.

A brief outline of the rest of the paper is organized as follows. In Section 2, the mathematical formulation for an inverse time-dependent source problem is briefly introduced. Section 3 presents the methodology of the GFDM and its numerical implementation for general 3D partial differential equations. The numerical strategies of the GFDM for inverse problems are also discussed. Next, in Section 4, several numerical examples involving both smooth and piecewise smooth geometries are presented. Finally, some conclusions and remarks are provided in Section 5.

2. Mathematical formulation for inverse time-dependent source problems

Consider a general three-dimensional homogeneous, isotropic domain Ω with boundary $\partial\Omega$ which was assumed to be sufficiently smooth in the sense of Liapunov. In this paper, we consider the following inverse heat source problem, to find a pair of functions ($u(x, y, z, t)$, $f(t)$), which satisfy the governing equation as follows:

$$\frac{\partial u(x,y,z,t)}{\partial t} = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} + f(t), \tag{1}$$

with the following initial and boundary conditions:

$$u(x,y,z,0) = u_0(x,y,z), \quad (x,y,z) \in \Omega, \tag{2}$$

$$u(x,y,z,t) = b(x,y,z,t), \quad (x,y,z) \in \partial\Omega, \quad t \in [0, t_{\max}]. \tag{3}$$

In the above Eqs. (1)–(3), $u_0(x,y,z)$ and $b(x, y, z, t)$ are given functions, while functions $u(x, y, z, t)$ and $f(t)$ are unknown. Problem (1)–(3) is ill-posed and in order to guarantee the uniqueness of the solution, as illustrated in Refs. [20,27], the following extra measured data at a fixed point $(x_0,y_0,z_0) \in \Omega$ should be given:

$$u(x_0,y_0,z_0,t) = h(t), \quad t \in [0, t_{\max}], \tag{4}$$

where $h(t)$ is a given function. The existence and uniqueness of solutions to such inverse problems have been studied in Ref. [18].

Let us define the following variable transformation [20,23]:

$$T(x,y,z,t) = u(x,y,z,t) - \int_0^t f(\xi)d\xi, \tag{5}$$

which transforms the original problem (1)–(4) into the following homogeneous partial differential equation:

$$\frac{\partial T(x,y,z,t)}{\partial t} = \frac{\partial^2 T(x,y,z,t)}{\partial x^2} + \frac{\partial^2 T(x,y,z,t)}{\partial y^2} + \frac{\partial^2 T(x,y,z,t)}{\partial z^2}, \tag{6}$$

with the following boundary/initial conditions:

$$T(x,y,z,0) = u_0(x,y,z), \quad (x,y,z) \in \Omega, \tag{7}$$

$$T(x,y,z,t) = b(x,y,z,t) - \int_0^t f(\xi)d\xi, \quad (x,y,z) \in \partial\Omega, \quad t \in [0, t_{\max}], \tag{8}$$

and the extra measured data at point $(x_0,y_0,z_0) \in \Omega$ become

$$T(x_0,y_0,z_0,t) = h(t) - \int_0^t f(\xi)d\xi, \quad t \in [0, t_{\max}]. \tag{9}$$

In the right-hand side of Eqs. (8) and (9), the function $f(\xi)$ is unknown. By substituting Eq. (8) into (9) for eliminating the function $f(\xi)$, it can obtainthe following equation:

$$T(x,y,z,t) - T(x_0,y_0,z_0,t) = b(x,y,z,t) - h(t),$$
$$(x,y,z) \in \partial\Omega, \quad t \in [0, t_{\max}]. \tag{10}$$

The numerical solutions of $T(x, y, z, t)$ can then be obtained by solving the above Eqs. (6), (7) and (10).

Let $r(t) = \int_0^t f(\xi)d\xi$ and substituting $T(x, y, z, t)$ into Eq. (5), we have

$$r(t) = \int_0^t f(\xi)d\xi = u(x_0,y_0,z_0,t) - T(x_0,y_0,z_0,t) = h(t) - T(x_0,y_0,z_0,t). \tag{11}$$

Using the procedure described above, the solutions of the original problem (1)–(3) are then given by

$$u(x,y,z,t) = T(x,y,z,t) + r(t), \tag{12}$$

and the unknown heat source $f(t)$ can be obtained as follows

$$f(t) = \frac{d\left(\int_0^t f(\xi)d\xi\right)}{dt} = \frac{dr(t)}{dt}. \tag{13}$$

It is noted that the solutions of Eq. (13) (the first derivative with respect to time) can be obtained numerically by using, for example, backward/forward Euler method, Crank–Nicolson method, and/or Runge–Kutta method. In our computations, this was achieved by using the central difference formula:

$$f(t_i) = \left.\frac{dr(t)}{dt}\right|_{t=t_i} = \frac{r(t_{i+1}) - r(t_{i-1})}{2\Delta t}. \tag{14}$$

3. The numerical method

Without loss of generality, let us consider the following 3D parabolic differential equation:

$$\frac{\partial T}{\partial t} = a_1\frac{\partial^2 T}{\partial x^2} + a_2\frac{\partial^2 T}{\partial y^2} + a_3\frac{\partial^2 T}{\partial z^2}, \tag{15}$$

or for brevity

$$\frac{\partial T}{\partial t} = L_2[T], \tag{16}$$

where $L_2[T]$ is a linear second-order partial differential operator. a_1, a_2, and a_3 are constants. The boundary and initial conditions for problem (15) are the same as these shown in Eqs. (2) and (3).

For the time-dependent problem considered here, we first separate the problem into, as in the finite element (FEM) and boundary element (BEM) methods, the space-domain and the time-domain parts. The proposed GFDM is then employed for discretization in the spatial domain and the second-order Crank–Nicolson scheme [49] for discretization in the time variable.

3.1. The GFDM for discretization in space-domain

First, we consider the GFDM for the numerical solution of the equation in the space variables, i.e., $L_2[T]$.

First of all, in order to obtain the explicit GFDM formulae for partial differential equations, an irregular cloud of points, as shown in Fig. 1, is scattered in the computational domain. For each given node $\boldsymbol{x}_0$ (central node), the m nearest nodes $\boldsymbol{x}_i$ ($i = 1, 2, ..., m$) (neighbors or support nodes) will be found within a prescribed distance d_m from the central node, $|\boldsymbol{x}_i - \boldsymbol{x}_0| \le d_m$. According to Gavete et al. [39], the concept of the "*star*" then refers to the area of support nodes in relation to the central node (see Fig. 1). Note that each node scattered inside the computational domain has an associated star assigned.

Suppose T_0 is the value of the function at the central node $\boldsymbol{x}_0$ and T_i ($i = 1, 2, ..., m$) are the function values at the rest of the nodes $\boldsymbol{x}_i$ inside

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