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Solution of multi-dimensional Klein–Gordon–Zakharov and Schrödinger/Gross–Pitaevskii equations via local Radial Basis Functions–Differential Quadrature (RBF–DQ) technique on non-rectangular computational domains

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A B S T R A C T

In the current investigation, we develop an efficient truly meshless technique for solving two models in optic and laser engineering i.e. Klein-Gordon-Zakharov and Schrödinger/Gross-Pitaevskii equations in one- two- and three-dimensional cases. The employed meshless is the upwind local radial basis functions-differential quadrature (LRBF-DQ) technique. The spacial direction is discretized using the LRBF-DQ method and also to obtain high-order numerical results, the fourth-order exponential time differencing Runge-Kutta method (ETDRK4) planned by Liang et al. [37] is applied to discrete the temporal direction. To show the efficiency of the proposed method, we solve the mentioned models on some complex shaped domains. Moreover, several examples are given and simulation results show the acceptable accuracy and efficiency of the proposed scheme.

1. Introduction

Recently meshless methods have been used for solving partial differential, integral and stochastic equations [34]. The meshless methods don't use any mesh, element or lattice to discrete the computational domain for obtaining some numerical results [22]. The mentioned property is a basic advantage for the meshless methods. The meshless methods can be classified in the two basic classes:

- The global form,
- The local form.

The meshless methods based on the global form can be applied for solving partial differential equations and integral equations, easily [2–4,22,35]. But these methods have some deficits for solving some PDEs such as advection equations and problems with blow up in solutions. In other hand, to overcome the mentioned deficiency the local meshless methods have been introduced. It should be noted that these techniques can be split in two forms [15,58]:

- Local meshless methods based on the variational (local) weak form,
- Local meshless methods based on the strong form.

In the first class, there are some integrals that must be computed thus these methods have more difficulty and need more CPU time. But the second class lacks any integral, thus is very flexible to solve all models with nonlinear term [61,62].

One of the local meshless collocation methods is the RBFs finite difference (RBFs-FD) method. The RBF-FD idea is developed in [23,27,28,43,44,47,50,51]. Authors of [26] developed a filter approach for RBF-FD that is related to traditional hyperviscosity and which can be applied quickly in any number of dimensions. Also, some analytical explanations related to the weights of Gaussian RBF-FD formula are obtained in [9]. The main aim of [7,8,10] is to obtain an optimal shape parameter for RBF-FD technique.

Recently, the RBF-FD has been employed and developed by researchers for example solving large-scale geoscience modeling [25], hyperbolic PDEs on the sphere [14] and diffusion- and also reaction-diffusion equations on closed surfaces [45]. Also see [8].

The differential quadrature method was first introduced by Bellman et al. [11]. The polynomial functions have been selected as the test function [46]. For the first time, authors of [47] proposed the meshless

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RBF-DQ method by using the RBFs. The RBF-DQ method is similar to the LRBF and RBF-FD methods.

The RBFs-DQ is employed for solving several PDEs such as equations in fluid dynamic [47,48], system of boundary value problems [20], coupled Klein–Gordon–Zakharov equations [21], doubly-curved shells made of composite materials [52], Stokes flow problem in a circular cavity [32], simulating natural convection in concentric annuli [59] etc.

The nonlinear partial differential equations (PDEs) play important role in modeling natural phenomena as many concepts in physics can be modeled by nonlinear PDEs [55,56]. In the current paper, we consider the following models:

1. The Schrödinger/Gross–Pitaevskii equation,
2. The Klein–Gordon–Zakharov (KGZ) equation.

1.1. Organization chart and the main aim of the manuscript

In the current paper, we employ the meshless local RBF-DQ method for solving the multi-dimensional Schrödinger/Gross–Pitaevskii and Klein–Gordon–Zakharov (KGZ) equations. To this end, we employ the meshless local RBF-DQ technique to discrete the spatial direction and a finite difference scheme for the temporal variable. We apply the proposed technique on some complex computational domains in two- and three-dimensional cases. As is well-known, the dispersion error related to a numerical technique has a direct effect for simulating the wave propagation phenomena. In other word, the wave frequency of the numerical solution and the wave frequency of the exact solution oppose and this difference can be increased when the frequency is increasing. In this case, the numerical techniques such as finite difference or finite element methods can not obtain accurate approximate solutions in limited to middle frequency range. In the current paper, we have proposed a numerical technique that it has suitable accuracy for the mentioned issue.

The structure of this article is as follows:

- *In Section 2, we explain the local radial basis function-differential quadrature (RBF-DQ) method.*
- *In Section 3, applying the local RBFs-DQ method on SGP equation is proposed.*
- *In Section 4, applying the local RBFs-DQ method on KGZ equation is developed.*
- *In Section 5, we report the numerical experiments of solving the considered models for some test problems.*
- *Finally, a brief conclusion of the current paper has been written in Section 6.*

1.2. The Klein–Gordon–Zakharov equation.

The d-dimensional Klein–Gordon–Zakharov equation is [5,12,39,60]

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} - 3v_0^2\Delta u(\mathbf{x},t) + \omega_p^2 u(\mathbf{x},t) + \omega_p^2 v(\mathbf{x},t)u(\mathbf{x},t) = 0, \tag{1.1}$$

$$\frac{\partial^2 v(\mathbf{x},t)}{\partial t^2} - c_s^2\Delta v(\mathbf{x},t) - \frac{n_0\epsilon_0}{4mN_0}\Delta\left(v^2(\mathbf{x},t)\right) = 0, \tag{1.2}$$

in which u and v are real-valued functions representing the fast time scale component of the electric field raised by electrons and the derivation of ion density from its equilibrium, respectively. Also, in the above relations:

- ω_p is the electron plasma frequency,
- c_s is the speed of sound,
- v_0 is the electron thermal velocity,
- n_0 is plasma charge number,
- ϵ_0 is vacuum dielectric constant,
- N_0 is electron density,
- m is ion mass.

For the most important applications of Eq. (1.1), we can mention :

- describing the mutual interaction between the Langmuir waves and ion acoustic waves in a plasma [13],
- adopted to model the strong Langmuir turbulence [13].

As is mentioned in [5], Eq. (1.1) can be derived from the two-fluid Euler-Maxwell system for the electrons, ions and electric field, by neglecting the magnetic field and further assuming that ions move much slower than electrons [5,12]. Under a proper nondimensionalization [39], the dimensionless Klein–Gordon–Zakharov system in d dimensions ($d = 1, 2, 3$) reads

$$\begin{cases} \varepsilon^2 u_{tt} - \Delta u + \dfrac{1}{\varepsilon^2}u + vu = 0, \\ v_{tt} - \Delta v - \Delta v^2 = 0, \end{cases} \quad \text{in } R^d, \tag{1.3}$$

where ε is a dimensionless parameter inversely proportional to the plasma frequency and is given by

$$0 < \varepsilon = \frac{\sqrt{3}v_0}{x_s\omega_p} = \frac{\omega_s}{\omega_p} \le 1,\ \omega_s = \frac{\sqrt{3}v_0}{x_s}.$$

1.3. The Schrödinger/Gross-Pitaevskii (SGP) equation

The generalized Gross-Pitaevskii equation is as follows:

$$iu_t + \beta\nabla^2 u + \gamma|u|^2u + Vu + W = 0, \quad \text{in}\,\Omega,\ t > 0, \tag{1.4}$$

with Dirichlet boundary condition

$$u(\mathbf{x},t) = g(\mathbf{x},t),\ \mathbf{x} \in \partial\Omega,\ t > 0, \tag{1.5}$$

and initial condition

$$u(\mathbf{x},0) = h(\mathbf{x}), \mathbf{x} \in \Omega. \tag{1.6}$$

The Gross–Pitaevskii equation is presented for the first time by Gross [29] and Pitaevskii [41] that describes the ground state of a quantum system of identical bosons using the Hartree-Fock approximation and the pseudo-potential interaction model [54]. The mentioned equation has been solved by different methods for example improving the classical variational approximation (VA) theory and applying the method of asymptotic analysis [38], a new lattice Boltzmann model for the interaction of two solitons [53], the adaptive grids based on wavelet method with time-splitting finite difference method [36], Kansa's approach and meshless local Petrov–Galerkin (MLPG) method [17], a Chebyshev pseudospectral multidomain method [18], a combination of boundary knote method (BKM) and meshless analog equation method (AEM) [19], spectral Fourier or spherical harmonics in the angular coordinates combined with generalised-Laguerre basis functions in the radial direction [42], a quantized vortex lattice dynamics in a rotating BEC [1], adaptive time-splitting schemes combined with fast Fourier transform techniques [49], etc.

2. The local RBFs-DQ method

At first, we present some explanations for radial basis functions.

2.1. A brief preliminaries for radial basis functions (RBFs) technique

In the current section, we explain the local RBF-DQ method thus at first we give some preliminaries for the radial basis function.

Definition 2.1. [24,57] A real valued continuous function $\phi \in \mathbb{R}^d \longrightarrow \mathbb{C}$ is positive definite if for all sets $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ of distinct points and all vectors $\lambda \in \mathbb{R}^d$