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Hybrid fundamental solution based finite element method for axisymmetric potential problems

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ABSTRACT

This paper describes a new type of hybrid fundamental solution based finite element method (HFS-FEM) for analysis of axisymmetric potential problems in multiply connected domain. In this approach, two independent potential fields are assumed within the element domain and on its boundary respectively. The fundamental solutions are utilized as internal trial functions to construct the non-conforming intra-element potential field. And the inter-element continuity is enforced by the conforming frame potential field which is of the same form as in the conventional FEM. Then, the axisymmetric modified variational functional is employed to derive the HFS finite element formulation. Finally, three numerical examples are given to demonstrate the validity, high-efficiency and robustness of the proposed method.

1. Introduction

The hybrid Trefftz finite element method (HT-FEM) [1,2] is widely reported in the literature, in which two independent assumed fields (non-conforming internal field and auxiliary conforming frame field) are usually employed. However, the drawback of the HT-FEM is that it is difficult to generate Trefftz functions for some physical problems and the terms of truncated T-complete functions should be carefully selected in achieving desired results. To remove the drawback, the hybrid fundamental solution based finite element method (HFS-FEM) has gained much attention in recent years [3–6]. In the method, the intra-element field in HFS-FEM is approximated by the linear combination of fundamental solutions at different points satisfying the corresponding governing equations, instead of T-complete functions adopted in HT-FEM. Adjacent elements are linked by inter-element boundary approximations constructed by the conventional nodal interpolation. A modified variational functional is established to enforce the inter-element continuity and derive the resultant element stiffness equation. The use of fundamental solutions can readily convert the domain integral in the modified variational functional to the boundary ones. To avoid the singular integrals inherited in fundamental solutions, all the source points are located outside each element as done in the method of fundamental solutions (MFS). Obviously, the HFS-FEM possesses all the advantages of HT-FEM and eliminates some of its drawbacks such as the intractability in establishing and selecting Trefftz functions [4].

The potential problems governed by Laplace equation appear in many scientific fields, like the heat conduction, seepage, corrosion, torsion problems and so on [7–13]. One of the earliest contributions to axisymmetric potential problems was conducted by Karageorghis and Fairweather [9] who employed the MFS to investigate the steady-state heat conduction and torsion. Recently, Smyrlis and Karageorghis [11] developed a meshless boundary collocation method for the solution of steady-state heat conduction in an isotropic bimaterial. In the past two decades, both HT-FEM and HFS-FEM have been successfully applied to two- dimensional (2D) and even three-dimensional (3D) potential problems. When the domain and boundary conditions are both axisymmetric, the original 3D problems usually reduce to solving the 2D cases with less computational time and memory. Wang et al. [14] developed a four-node hybrid Trefftz annular element for analyzing the axisymmetric potential problems. The internal potential field is approximated by the suitably truncated quasi-harmonic polynomials and the annular element is immune to the mesh distortion. To the authors' knowledge, few reports on the HFS-FEM for analyzing axisymmetric potential problems are available in the literature. Wang and Qin [15] investigated the axisymmetric thermal behavior of composites enhanced with carbon nanofibers or nanotubes by using the HFS-FEM and cylindrical representative volume element (RVE).

In the current study, an eight-node quadrilateral HFS annular element is proposed for investigating the axisymmetric potential problems in the multiply connected domain. A brief outline of this paper is listed as follows. In Section 2, the basic theory and formulations of HFS-FEM are presented through a simple description of axisymmetric potential problems. The performance of HFS-FEM is numerically assessed in Section 3 in comparison with the analytical solutions and conventional

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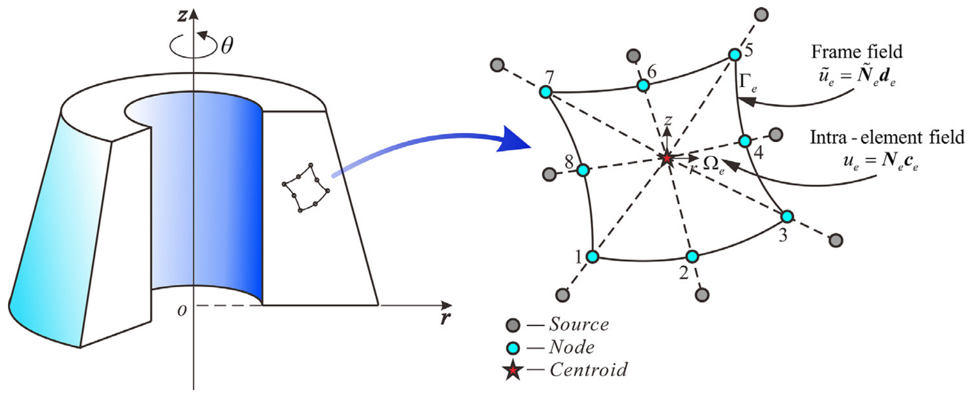

Fig. 1. An eight-node annular element for the axisymmetric potential problems.

finite element (ABAQUS) results. Concluding remarks and possible extensions are discussed in Section 4.

2. Theory

2.1. Governing equations

Let us consider the following 3D potential problem

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial u}{\partial z}\right)=0 \text{ in } \Omega^* \tag{1}$$

subject to the boundary conditions including Dirichlet boundary condition

$$u(x,y,z)=\bar{u}(x,y,z) \quad \text{on } \Gamma_u^* \tag{2}$$

and Neumann boundary condition

$$q(x,y,z)=\lambda\frac{\partial u}{\partial x}n_x+\lambda\frac{\partial u}{\partial y}n_y+\lambda\frac{\partial u}{\partial z}n_z=\bar{q}(x,y,z) \text{ on } \Gamma_q^* \tag{3}$$

where λ stands for the property coefficient and Ω^* denotes a bounded domain in $\Re^3$ space with boundary Γ^* ($\Gamma^*=\Gamma_u^*\cup\Gamma_q^*$). Additionally, n_x, n_y and n_z represent direction cosines of the outward normal at a given boundary point. The region $\Omega^*\in\Re^3$ is axisymmetric, that is, formed as a geometry of revolution by rotating a planar domain Ω with boundary Γ about the z axis. When the boundary conditions are also axisymmetric, it is obvious that both u and its normal derivative q on the boundary Γ are independent of the azimuthal angle θ [16]. For simplicity, the 3D potential problem will reduce to solving an axisymmetric version in the cylindrical coordinates r and z:

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \text{ in } \Omega \tag{4}$$

together with the boundary conditions

$$u(r,z)=\bar{u}(r,z) \text{ on } \Gamma_u \tag{5}$$

and

$$q(r,z)=\lambda\frac{\partial u}{\partial r}n_r+\lambda\frac{\partial u}{\partial z}n_z=\bar{q}(r,z) \text{ on } \Gamma_q \tag{6}$$

where ∇^2 denotes the Laplace operator and $\Gamma=\Gamma_u\cup\Gamma_q$.

2.2. Assumed fields and formulation of HFS-FEM

2.2.1. Non-conforming intra-element potential field

Two independent assumed potential fields are employed over each element, say element e, which occupies the sub-domain Ω_e. To eliminate the singularities of fundamental solutions, the field variable defined in the element domain is expressed by a linear combination of fundamental solutions centered at different sources located outside the element [4] (see Fig. 1). Thus, the non-conforming intra-element potential field is given by

$$u_e(\mathbf{Q})=\sum_{j=1}^{n_s}N_e(\mathbf{P}_j,\mathbf{Q})c_{ej}=\mathbf{N}_e(\mathbf{Q})\mathbf{c}_e \qquad \forall\mathbf{Q}\in\Omega_e,\mathbf{P}_j\notin\Omega_e \tag{7}$$

where c_{ej} are undetermined coefficients and n_s is the number of virtual sources outside the element. The fundamental solution of Eq. (7), $N_e(\mathbf{P}_j,\mathbf{Q})$, usually satisfies

$$\nabla^2 N_e(\mathbf{P},\mathbf{Q})+\delta(\mathbf{P},\mathbf{Q})=0 \qquad \forall\mathbf{P},\mathbf{Q}\in\Re^2 \tag{8}$$

which gives

$$N_e(\mathbf{P},\mathbf{Q})=\frac{4K(k)}{R} \tag{9}$$

where $\mathbf{P}=\{r_P,\ z_P\}^{\mathrm{T}}$ is the virtual source point outside the element, $\mathbf{Q}=\{r_Q,\ z_Q\}^{\mathrm{T}}$ is the field point within the element domain, $R^2=(r_Q+r_P)^2+(z_Q-z_P)^2$, $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k)=\int_0^{\pi/2}\left[1-k^2\sin^2\varphi(\mathbf{Q})\right]^{-1/2}d\varphi(\mathbf{Q}) \tag{10}$$

and here

$$k^2=4r_Qr_P/R^2 \tag{11}$$

In the implementation, the location of sources are usually generated by means of the following relation employed in the method of fundamental solutions [17,18]:

$$\mathbf{P}=\mathbf{Q}_b+\gamma\left(\mathbf{Q}_b-\mathbf{Q}_c\right) \tag{12}$$

where γ is a dimensionless parameter, $\mathbf{Q}_b$ is the elemental boundary point (the nodal and/or the middle nodes of the element in the current study), and $\mathbf{Q}_c$ is the centroid of the element. In case the source points are too close to the element boundary, the solution is not accurate while in case they are too far away from the element boundary, the discretization matrix becomes ill-conditioned. Therefore, the optimal value of γ should be determined by numerical examples. Fig. 1 displays the sources which are generated by the nodes of the element.

The corresponding outward normal derivative of u_e on Γ_e may be expressed as

$$q_e=\sum_{j=1}^{n_s}\Theta_e(\mathbf{P}_j,\mathbf{Q})c_{ej}=\mathbf{\Theta}_e\mathbf{c}_e \tag{13}$$

where

$$\mathbf{\Theta}_e=\mathbf{A}\mathbf{T}_e \tag{14}$$

with

$$\mathbf{A}=\left[\lambda n_r \quad \lambda n_z\right] \quad \text{and} \quad \mathbf{T}_e=\left[\frac{\partial\mathbf{N}_e}{\partial r} \quad \frac{\partial\mathbf{N}_e}{\partial z}\right]^T \tag{15}$$