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Using radial basis functions to solve two dimensional linear stochastic integral equations on non-rectangular domains

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ABSTRACT

The main goal of this paper is presenting an efficient numerical scheme to solve two dimensional linear stochastic integral equations on non-rectangular domains. The proposed method is based on combination of radial basis functions (RBFs) interpolation and Gauss–Legendre quadrature rule for double integrals. The most important advantage of proposed method is that it does not require any discretization and so it is independent of the geometry of the domains. Thus, many problems on the irregular domains can be solved. By using this method, the solution of consideration problem is converted to the solution of the linear system of algebraic equations which can be solved by a suitable numerical method. Also, the convergence analysis of this approach is discussed. Finally, applicability of the present method is investigated through illustrative examples.

1. Introduction

Consider the following two dimensional linear stochastic integral equation of the second kind

$$f(x,y) = g(x,y) + \int\int_D k_1(x,y,s,t) f(s,t)\,dt\,ds + \int\int_D k_2(x,y,s,t) f(s,t)\,dB(t)\,dB(s), \quad (x,y) \in D, \tag{1}$$

where the kernels k_1 and k_2 and function g are known functions, f is unknown function which should be determined, $B(t)$ is standard Brownian motion process defined on probability space (Ω, A, P) consisting of the sample space Ω, a σ-algebra A of subsets of Ω called events, and a real-valued set function P defined on A called a probability. Also, $D \subseteq \mathbb{R}^2$ is a two dimensional non-rectangular domain and $\int\int_D k_2(x, y, s, t) f(s, t) dB(t) dB(s)$ is the double Itô integral. We assume that the functions g, k_1 and k_2 are continuous functions to ensure well-posedness of Eq. (1).

In recent decade by increasing computational power, it becomes essential to utilize more accurate mathematical equations such as stochastic functional equations for modeling problems occur in real life. Studying the behavior of stochastic integral equations has attracted the attention of many researchers. This tendency is due to a wide variety of applications in finance, biology, engineering and physics such as the stochastic formulation of problems in reactor dynamics [1], the study of the growth of biological population [2], the model of rainfall–runoff [3] and etc. The main obstacle for solving two dimensional stochastic integral equations is computational complexity of mathematical operations due to the randomness. Also in more cases, the analytical solution of these equations is not available or finding their analytical solution is very difficult. Thus, presenting an accurate and efficient numerical method is an essential requirement. There exist some papers on developing numerical approach to solve various type of stochastic functional equations such as stochastic Itô–Volterra integral equations [4–7], stochastic integro-differential equations [8–10], Stratonovich Volterra integral equations [11–13].

Two dimensional stochastic integral equations have significant application in various kind of science and engineering such as nonhomogeneous elasticity and electrostatics, the Dorboux problem, contact problems for bodies with complex properties, radio wave propagation, the elastic problem of axial translation of a rigid elliptical disk-inclusion, various physical and mechanical and biological problems. Despite wide applications, little studies have been done on numerical solution of multidimensional stochastic integral equations. M. Khodabin et al. used Haar wavelet for solving two-dimensional linear stochastic Fredholm integral equations [14]. Also, M. Fallahpour et al. presented a numerical scheme to solve two-dimensional linear stochastic Volterra–Fredholm integral equations via block-pulse functions [15]. Moreover, F. Mirzaee and E. Hadadian [16] applied two dimensional modification of hat functions to solve two dimensional Stratonovich Volterra integral equations. But, there exist still very few works on numerical solution of two dimensional stochastic integral equations.

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Recently, application of RBFs has changed from scattered data interpolation to the numerical solution of partial differential equations or integral equations. Meshfree methods based on RBFs were extended to solve different ordinary differential equations, partial differential equations and integral equations. For example RBFs have been applied to solve non-local parabolic partial differential equations [17], Klein–Gordon equation [18], time-domain electromagnetics [19], shallow water equations [20], acoustic wave propagation problems [21], heat transform problem [22], nonlinear Fredholm integral equation [23], nonlinear Volterra–Fredholm–Hammerstein integral equations [24], two dimensional Fredholm integral equations [25], multidimensional linear Fredholm integral equations [26], simulation rotative electromagnetic machines [27] and electrostatic problems [28].

In this paper, we use interpolation method via positive definite functions such as Gaussian RBFs and Multiquadric RBFs to solve two dimensional linear stochastic integral equations on non-rectangular domains. The reminder of this work is organized as follows. In Section 2.1, we obtain some elementary definitions and properties of stochastic processes such as the definition of Brownian motion process and some properties which are necessary to understand the papers in this field. Scattered data interpolation in high dimensions by using RBFs is explained in Section 2.2. In Section 3, the RBFs approximation is used to estimate the solution of two dimensional linear stochastic integral equations on non-rectangular domains. In Section 4, convergence analysis of the proposed method is proved. Numerical examples are included in Section 5. Finally, we give the conclusion of this paper in Section 6.

2. Preliminaries

In this section, we review some essential definitions and mathematical preliminaries about stochastic calculus and meshfree method which are applied in the current paper.

2.1. Stochastic calculus

A stochastic process is a phenomenon which evolves in time in a random way. There are an enormous variety of phenomena in nature which can be considered as a function both of time and of a random factor. For instance the price of certain commodities, the size of some populations, RLCG cell [29] and the number of particles registered by a Geiger counter. A well-known process in the calculus of continuous process is the Brownian motion process. This phenomenon is named due to its discovery by the English botanist R. Brown in 1827. The physical theory of this motion was set up by Einstein in 1905. It suggests that this motion is random and has the following properties:

(i) it has independent increments;
(ii) the increments have Normal distribution;
(iii) the motion is a continuous function of t.

So, the formal definition of Brownian motion process is as follows.

Definition 1. Brownian motion $\{B(t)\}$ is a stochastic process which satisfies in the following properties [30]

1. $B(t)-B(s)$ for $t>s$ is independent of the past. That means for $0<u<v<s<t<T$, the increments $B(t)-B(s)$ and $B(v)-B(u)$ are independent.
2. $B(t)-B(s)$ for $t>s$ has Normal distribution with mean zero and variance $t-s$. In other words, $B(t)-B(s)\sim\sqrt{t-s}N(0,1)$ where $N(0, 1)$ denotes Normal distribution with zero mean and unit variance.
3. $B(t)$, $t\geq 0$ are continuous functions of t.

Five realizations of Brownian motion $B(t)$ are plotted in Fig. 1.

Let $g(x)=|x|^p, p\geq 1$. The collection of all random variables X which $Y=g(X)$ has finite expectation forms a complete normed linear space (Banach space). This space is denoted by $L^p(\Omega, A, P)$ and the norm in this space is defined as [31]

$$\|X\|_p=\{E(|X|^p)\}^{\frac{1}{p}}. \tag{2}$$

The stochastic integral $\int_0^T f(t)dB(t)$ is called the Itô integral. The Itô integral can be approximated as

$$\int_0^T f(t)dB(t)\simeq\sum_{i=0}^{N-1} f(t_i)(B(t_{i+1})-B(t_i)), \tag{3}$$

where $t_i=idt$, $dt=\frac{T}{N}$ and N is a sufficiently large number.

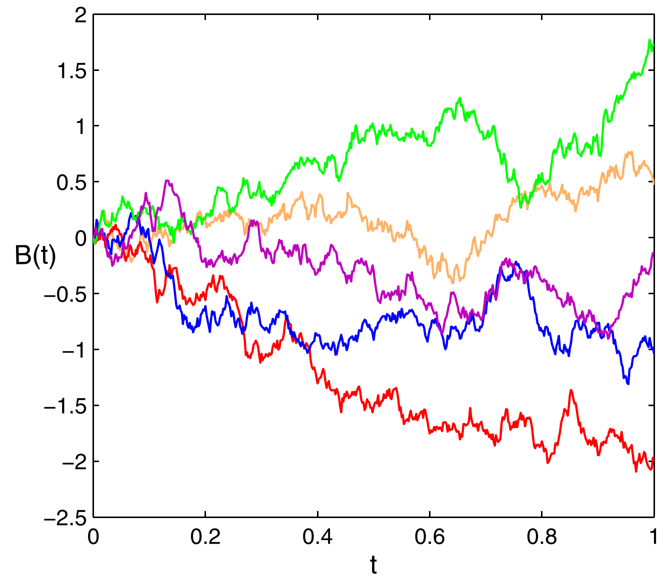

Fig. 1. Five paths of Brownian motion $B(t)$.

Table 1
Definitions of some well-known RBFs.

Name of RBFs	Definition
Gaussian (GA)	$\phi(r)=\exp(-\epsilon r^2)$
Multiquadrics (MQ)	$\phi(r)=\sqrt{r^2+\epsilon^2}$
Inverse multiquadrics (IMQ)	$\phi(r)=(\sqrt{r^2+\epsilon^2})^{-1}$
Thin plate spline (TPS)	$\phi(r)=(-1)^{k+1}r^{2k}\log(r), k\in\mathbb{N}$
Inverse quartics (IQ)	$\phi(r)=(r^2+\epsilon^2)^{-1}$

2.2. Meshfree method based on RBFs

Meshfree methods have gained much attention of mathematicians and engineers in recent years. Thus, there exist numerous works concerned with meshfree approximation methods in different branches of science. Moreover, computation with high dimensional data is an important issue in many areas of science and engineering. Many traditional numerical methods such as finite difference method and finite element method can either not handle such problems at all, or are limited to very special situations. So, RBFs are extensively applied to approximate multivariate functions or scattered data interpolation in high dimensional.

Definition 2. A function $\Phi:\mathbb{R}^s\to\mathbb{R}$ is called radial provided there exists a univariate function $\phi:[0,\infty)\to\mathbb{R}$ such that [32]

$$\Phi(\mathbf{x})=\phi(r), \tag{4}$$

where $r=\|\mathbf{x}\|$ and $\|.\|$ is usually the Euclidean norm in $\mathbb{R}^s$.

Some popular RBFs with shape parameter ϵ are listed in Table 1 and are plotted in Fig. 2.