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A LBIE-RBF solution to the convected wave equation for flow acoustics

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A B S T R A C T

In this paper, the meshless local boundary integral equation (LBIE) method is implemented for the solution of convected wave equation in frequency domain. Such a problem plays an important role for the noise prediction in flow acoustics. The case of a uniform flow in x-direction is analyzed. For the resultant convected Helmholtz equation, the free-space Green's function has a more complicated form than the Green's function for the classical Helmholtz equation, though it reduces to the latter when a coordinate (Prandtl–Glauert) transformation is applied. Furthermore, the LBIE method requires the determination of the solution of the corresponding Dirichlet's problem in local subdomains (companion solution). Therefore, the relevant companion solution is derived. Radial basis functions are used for the interpolations of the field variable. Numerical examples are shown for cases up to frequency of 5000 Hz and Mach number of 0.5, and the results are compared with FEM software (COMSOL).

1. Introduction

The solution of the convected wave equation is an important task in many aeroacoustics applications, such as the prediction of rocket noise [1], the use of acoustic metamaterials [2], turbofan aircraft engine [3] and compressor noise [4,5]. Several numerical methods have been developed in the literature for the solution of the convected wave equation. In [6, 7], the stability and accuracy of the finite element methods (FEM) was investigated by means of the dispersion error for 1D and 2D problems. Adapted spectral FEM [8], subgrid scale FEM [4] and high-order FEM [9] versions have been also implemented previously. As mentioned in [6,7,8], the numerical methods suffer from errors because of the spurious modes that appear, the dispersion error, and the different wavenumbers of the outgoing and incoming waves.

Regarding boundary element methods (BEM), the direct boundary integral formulation and the expression for the free space Green's function was presented by Wu and Lee [10]. Lee and Lee [11] have shown the derivation and the implementation of the direct BEM in time domain. In Ref. [12], the formulations in [10] were extended to 3D dimensional problems with non-uniform flow, where the non-uniform contributions in the domain were handled using the dual-reciprocity method. Coupled FEM-BEM approach can be also implemented as in Balin et al. [13], where the calculations in the potential flow zone are performed with the FEM and the uniform flow zone is solved via BEM.

The local boundary integral equation (LBIE) method is a well-established meshless numerical approach which has been previously used for the solution of potential, electromagnetic, diffusion, elasticity, and wave problems [14,15]. It has a similar foundation as the domain-decomposition based, multi-domain BEM (MD-BEM) [16]. However, in contrast to the latter, the subdomains in the LBIE may be overlapping. Furthermore, note that the full-matching boundary conditions at the subdomain boundaries are implemented in the MD-BEM, whereas boundary and domain integrals are computed for the overlapping subdomains in the LBIE, and the unknown variables at the quadrature integration points are interpolated from the neighboring source nodes with the use of Radial Basis Functions (RBFs) or the use of Moving Least Squares. Chen et al. [17] has shown that, when solving the Helmholtz equation with the LBIE method, the use of frequency-dependent functions in the interpolations provides more accurate results than the usual polynomial based functions. The dispersion error of the LBIE for a 2D Helmholtz problem was also investigated by Dogan et al. [18]; it has been found that its accuracy is comparable to the classical FEM with linear elements.

In the present paper, the implementation of the LBIE method for the convected wave equation is shown for the case of 1D uniform flow. The frequency domain convected wave equation in Ref. [10] is taken as the basis. A crucial step for the local BEM is the derivation of the companion solution in order to eliminate the derivative of the velocity potential from the formulations; this has been accomplished. Moreover, the variables are expressed in the physical space rather than the transformed space. No transformation is required for the boundary conditions, unlike the methods based on transformed formulations [13]. The establishment of such a method is an important step toward the solution of the acoustic propagation in three dimensional complex flows. Because

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the integral equation is written for sufficiently small subdomains and the free space Green's function is applied locally in the present method, the formulations can be easily extended to 3D and non-uniform cases.

Although volume based methods such as FEM and finite difference scheme are able to solve wave propagation in a non-uniform medium, fewer research has been devoted to the convected wave propagation with non-uniform Mach number distribution using the BEM [19]. As discussed recently by Mancini et al. [19], the existing works [12,19,20,21] have treated the problem with global boundary integral equations; they have provided only approximate solutions or applied dual reciprocity technique, the latter of which suffers from the definition of robust interpolation functions. Appropriate integral kernels for non-uniform case are yet to be derived for the BEM, since the free space Green's function as used in the direct boundary integral formulation cannot handle the non-uniformities in the domain. The technique of decomposing the Mach number into a uniform component and small deviations has been employed, though the results deteriorate with increasing frequency and mean flow Mach number [19,20]. These features show a potential for the analysis of the problem with local integral equations and/or meshless methods. As such, the LBIE method presented has no such restrictions. The formulations are valid for all parameter ranges with Mach number $M<1$. Hence, the strongly non-uniform effects observed in the flow calculations can be inserted into the acoustic computations. The RBF interpolation matrix is already constructed in the LBIE for each subdomain for the interpolations of the field variable (e.g. the velocity potential). Therefore, the interpolations of the Mach number in case of a non-uniform flow would not bring in any additional cost in terms of computational time and global error of the method.

2. Formulations

Let us consider the frequency domain acoustic wave equation in a steady uniform flow

$$\nabla^2\Phi(\boldsymbol{x}) + k^2\Phi(\boldsymbol{x}) - 2ikM\frac{\partial\Phi(\boldsymbol{x})}{\partial x} - M^2\frac{\partial^2\Phi(\boldsymbol{x})}{\partial x^2} = 0, \quad \boldsymbol{x}\in V, \tag{1}$$

where Φ is the velocity potential, k is the wavenumber, M is the Mach number of the uniform flow in the positive x direction, and $i=\sqrt{-1}$. The domain V is enclosed by $\Gamma=\Gamma_u\cup\Gamma_q$ with the boundary conditions

$$\Phi(\boldsymbol{x}) = \Phi_0, \ \text{on}\ \Gamma_u, \tag{2a}$$

$$\frac{\partial\Phi(\boldsymbol{x})}{\partial n} \equiv q = q_0, \ \text{on}\ \Gamma_q, \tag{2b}$$

where Φ_0 is the prescribed velocity potential on the essential boundary Γ_u, q_0 is the prescribed potential flux on the boundary Γ_q, and n is the outward normal vector on Γ.

The Green's function of the adjoint operator of Eq. (1) was derived in Ref. [10], and is given by

$$G(\boldsymbol{x}) = \frac{e^{-ik\frac{\sqrt{x^2+(1-M^2)(y^2+z^2)}+Mx}{1-M^2}}}{4\pi\sqrt{x^2+\left(1-M^2\right)\left(y^2+z^2\right)}}. \tag{3}$$

A weak formulation of the problem may be formed using the test function G such that

$$\int_V G\left(\nabla^2\Phi + k^2\Phi - 2ikM\frac{\partial\Phi}{\partial x} - M^2\frac{\partial^2\Phi}{\partial x^2}\right)dV = 0, \tag{4}$$

which yields the integral equation representation:

$$\lambda\,\Phi(P) = \int_\Gamma \left[\left(G\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial G}{\partial n}\right) - 2ikMG\Phi n_x - M^2\left(G\frac{\partial\Phi}{\partial x} - \Phi\frac{\partial G}{\partial x}\right)n_x\right]d\Gamma, \tag{5}$$

where n_x is the x component of the unit normal vector n.

In the LBIE, a set of (N) source nodes are distributed in the domain and on the boundary Γ. Spherical subdomains are generated, such that $\Omega\in V$. The surface (boundary) enclosing each interior subdomain is defined as $\Omega_S=\partial\Omega$ (see Fig. 1). When the source node lies on the global boundary, the subdomain is not a full sphere anymore; it takes the shape of a slice of a sphere the azimuthal and polar angle bounds of which need to be determined. For such a case, $\partial\Omega=\Omega_b\cup\Omega_S$, where $\Omega_b\in\Gamma$ and $\Omega_S\in V$. The integral Eq. (5) must hold then for all subdomains. Assuming that the collocation point P is located at the center of the subdomain, we write

$$\lambda(P)\,\Phi(P) = \int_{\partial\Omega}\left[\left(G\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial G}{\partial n}\right) - 2ikMG\Phi n_x - M^2\left(G\frac{\partial\Phi}{\partial x} - \Phi\frac{\partial G}{\partial x}\right)n_x\right]d\Omega_S. \tag{6}$$

where

$$\lambda(P) = \int_{\partial\Omega}\left[\frac{\partial G_0}{\partial n} - M^2\frac{\partial G_0}{\partial x}n_x\right]d\Omega_S, \tag{7}$$

with $G_0(\boldsymbol{x}) = 1/4\pi\sqrt{x^2+(1-M^2)(y^2+z^2)}$.

Eq. (7) is still not adequate to be implemented because the term $\partial\Phi/\partial n$ is unknown on all of the boundaries of the subdomains. In the classical BEM, it would be known on some part of the global boundary (e.g. Neumann condition), and would be solved for on the rest of the global boundary where it is unknown. In meshless methods, one way to handle the unknown $\partial\Phi/\partial n$ on the subdomain boundaries is to express it in terms of spatial derivatives $\partial\Phi/\partial x$, $\partial\Phi/\partial y$ and $\partial\Phi/\partial z$. By doing so, four equations for each subdomain would be formulated for a 3D problem, and the resultant system matrix would have a size of $4N\times 4N$ (in a domain with N source nodes). In fact, in Ref. [22] such a method was developed for the solution of the Helmholtz equation.

We will proceed instead with one of the main features of the LBIE method, i.e. the elimination of the term $\partial\Phi/\partial n$ from the equations by using the companion solution technique. For instance, the companion solution within the frame of LBIE was applied for the solution of the Laplace problem in Ref. [14], and for the solution of the Helmholtz problem in Refs. [15,18,23]. For the convected wave equation, the companion solution needs to be formulated, which will be done in the following sections.

2.1. Transformed domain variables

In Ref. [10], the Prandtl–Glauert transformation for the acoustic radiation in a subsonic uniform flow was introduced, which is required in the conventional boundary integral formulation. Although it will be used for limited purposes in the present paper, it is beneficial to introduce such a transformation and to list the expressions in the transformed domain. For the spatial coordinates,

$$\tilde{x} = x/\sqrt{1-M^2}, \quad \tilde{y} = y, \quad \tilde{z} = z, \tag{8}$$

hence

$$\tilde{r} = r_M \,/\, \sqrt{1-M^2}. \tag{9}$$

The wavenumber in the transformed domain is defined as $\tilde{k} = k\sqrt{1-M^2}$, though an additional variable that is proportional to the wavenumber is also used, e.g. $\kappa = kM/\sqrt{1-M^2}$. The velocity potential can be transformed by imposing $\tilde{\Phi} = \Phi\, e^{-i\kappa\tilde{x}}$. Then, Eq. (1) reduces to the Helmholtz equation in the transformed domain (see [10] for details). Furthermore, the Green's function in the transformed domain is given by

$$\tilde{G} = G\, e^{i\kappa\tilde{x}} = \frac{1}{\sqrt{1-M^2}}\frac{e^{-i\tilde{k}\tilde{r}}}{4\pi\tilde{r}}. \tag{10}$$

2.2. The companion solution

The aim of the companion solution is to introduce a function the value of which is equal to the Green's function on the boundary of the subdomain [24]. Such a choice aids the use of a test function in the form $(G-G^*)$ in the weighted residual integral (4), which then provides the