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An improved regularized fundamental solution to the 2-D steady-state Stokes equation

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ABSTRACT

A new regularized fundamental solution to the Stokes equation in 2-D notation was derived using the new cut-off (blob) function. It substitutes the delta-Dirac function, which represents source term in a classical form of the equation expressed with the Green's functions. The proposed cut-off function/blob selection led to a much simpler and, therefore, faster fundamental solution in terms of elapsed computational time. Thus, it was possible to find boundary values and solve the linear system of the Stokes equations in 114 s applying backslash solver in Matlab 2013 on a quad-core 2.8 GHz each and 8RAM processor machine. This computation time appeared to become approximately 2.5 times less than the time required to run the same code with the known regularized fundamental solution to the 2-D steady-state Stokes equations.

Validation of the proposed solution was performed using well-known problems of the Stokes flow in a lid-driven cavity, as well as in a 2-D rectangular channel with semi-circular and parabolic protrusions used by Gaver, Kute and Cortez. The solutions for the normal and shear stresses, velocity magnitude and streamlines were obtained and compared to the corresponding solutions of Gaver, Kute, Cortez and the finite-volume method.

1. Introduction

In the last decade, the boundary element methods (BEM) have become more noticeable for modeling and numerical simulations along with the fast development of the microfluidics. Designing of the labs-on-chip, fine mixing devices, microchannel heat exchangers, fibrous filtration and particulate flows are just a few examples of the contemporary physics where BEM application is possible.

The "lab-on-a-chip" devices employing electrophoretic effects and microfluidics are very popular in experimental biomechanics [1] and pharmaceutics [2,3]. Because of the demand for extreme reliability of such devices, proper modeling and numerical simulations to predict physical effects mentioned above need to be performed. The commercial software based on finite element (FEM) and finite volume (FVM) methods can be numerically expensive in terms of meshing and solving systems of equations for entire domains. The boundary element methods can be helpful since they only require conditions set at a boundary of a considered domain for steady-state problems. Thus, BEM are more advantageous for a certain class of problems in microfluidics.

For example, in [4] Chowdhury et al. discuss application of the BEM to analyze electromagnetic and fluid-flow systems. They applied BEM to predict traction forces exerted at the droplet surface as well as to evaluate the flow field and traction forces inside an arbitrary channel.

The BEM discussed in the present paper and applied to the solution of the Stokes flow problems is the Boundary Singularity Method (BSM) [5–7], which can also be referred to as the Method of Fundamental Solutions (MFS). BSM requires allocation of source points (or Stokeslets) at the boundaries of a considered domain. Another set of points matching the number of Stokeslets – collocation points – is located inside the domain. In addition, a 1-D vector of boundary conditions C required to calculate boundary values or so-called Stokeslet strengths is used.

However, despite the advantages of the BEM applied to the micro- and nano-scale fluid flow problems, there are well-known singularity effects and the solution divergence issues at a considered boundary [5]. This is due to the nature of the Poisson's equation fundamental solution where external forces at a boundary induced by the flow are represented with the singular delta-Dirac function. These equations (1) are conveniently solved using the Boundary Element Methods and applying specific Stokeslets' placement [5–7] and regularization techniques [8,9] that have been implemented to avoid singular solutions at the boundaries [5–10] for the fluid dynamics problems satisfying $Re < 1$ condition.

The Stokes equation in the form of a Green's function solution can be written as:

$$\nabla^2 G(r-\xi) - grad(P) = \delta(r-\xi), \tag{1.1}$$

where G is a Green's function, P is a pressure solution, r and ξ are coordinate vectors in 2-D polar coordinate system associated with the source points (Stokeslets) and collocation points respectively. It is easy to notice that the right-hand side of Eq. (1.1) has a singularity at $|r-\xi|$. The

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delta-Dirac function $\delta(r-\xi) \to \infty$ as $|r-\xi| \to 0$, which corresponds to the boundary of a considered flow domain. In order to avoid singularity at the boundaries and suppress solution oscillations, a regularization technique was developed and introduced by Cortez et al. in [8–10]. As the result, Eq. (1.1) was in fact replaced by the following one:

$$\nabla^2 G_\epsilon(r-\xi) - grad\left(P_\epsilon\right) = \phi_\epsilon(r-\xi), \tag{1.2}$$

where G_ϵ and P_ϵ are the regularized equivalents of a Green's function and a pressure solution, respectively. The regularization of the governing equation above is introduced with a so-called blob or cut-off function $\phi_\epsilon(r-\xi)$. The selection criterion is discussed in [10], where a few examples of blobs and cut-off functions are introduced for both 2-D and 3-D notations.

The present paper is constructed as follows: Section 2 provides with the description of the alternative cut-off function, as well as with a definition of the modified radius-vector containing regularization parameter to avoid singularity effects at boundaries. The cut-off function is then introduced in the Green's function solution formulation.

Section 3 is dedicated to the validation of the proposed solution with three well-known Stokes flow problems. The first problem describes the Stokes flow in a 2-D rectangular channel with semi-circular protrusion. The solution of the problem represents normal and shear stress components distribution along the lower channel surface with the protrusion, as well as the velocity magnitude contours and the streamlines. The regularized fundamental solutions for the stress components have been compared to those obtained by Cortez in [10] and with the Gaver–Kute solution in [14] for the same problem. The second validation problem is a modification of the first one, where semi-circular protrusion is replaced with a parabolic obstacle. It is formulated as the Stokes flow over a parabolic obstacle inside a square domain, similarly to the problem in Chapter VIII of the [15]. The validation case represents a well-known problem of a Stokes flow in a lid-driven cavity [11–13]. In the last two cases, the proposed numerical solution results are compared to the ones obtained with the regularized Stokeslet presented in [10].

Finally, Section 4 concludes the paper and introduces the results of the comparison of the elapsed time to solve the matrix Eq. (1.3) for the proposed regularized solution with the one presented in [10] for 2-D class of Stokes flow problems:

$$F = C \backslash M, \tag{1.3}$$

where F is a vector of Stokeslets' strength components, which is used to derive velocity and pressure solution components, C is a vector of velocity boundary conditions having $3N$ elements and M is a $3N\times 3N$ computational matrix consisting of paired distances between Stokeslets and collocation points at the boundary using Einstein summation [5–7]. Here, N is the number of singularities (Stokeslets) that are placed over a considered domain. The resulting solution vector is represented as follows:

$$U = C - M_u F, \quad P = M_p F \tag{1.4}$$

where U is the velocity vector having the same dimension as C. The velocity and pressure solution components are introduced in the next Section.

2. Derivation of the regularized fundamental solution

In the present section, a cut-off function/blob, which is different from the one presented in [10]. A new cut-off function is aimed at simplification of the resulting regularized fundamental solution of the 2-D Stokes equation.

It is worthwhile introducing a so-called modified radius-vector ϱ, which incorporates regularization parameter ϵ:

$$\varrho = \sqrt{|r|^2 + \epsilon^2} + \epsilon, \tag{2.3}$$

where r is a radius-vector between Stokeslets and collocation points.

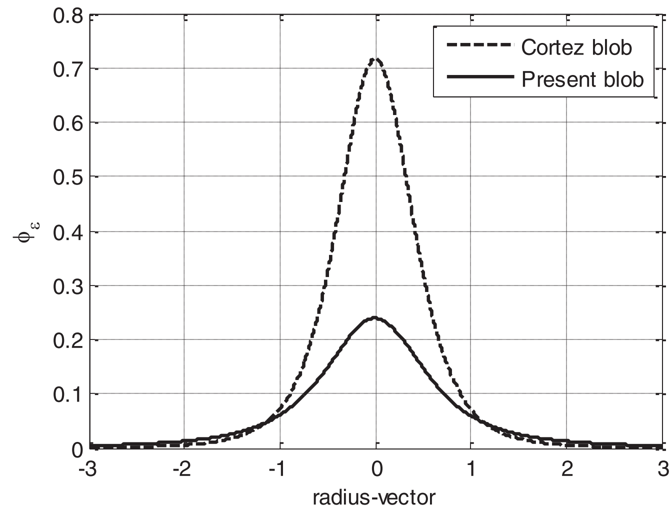

Fig. 2.1. Cut-off function distribution for $\epsilon = 0.816$: a) Cortez blob as per [10] and b) introduced blob.

The following cut-off function was selected in order to find corresponding Green's function solution kernel for the regularized 2-D Stokes equation fundamental solution:

$$\phi_\epsilon(r) = \frac{\epsilon}{2\pi\left(r^2+\epsilon^2\right)^{\frac{3}{2}}}. \tag{2.4}$$

The specified blob or cut-off function is presented and compared to the one introduced in [10] by Cortez for 2-D Stokes equation is presented in Fig. 2.1.

To check the validity of the selected function to the one introduced in [10], we integrate each of them over r in semi-infinite domain:

$$\begin{aligned}\int_\Omega \phi_\epsilon(r)dr &= \int_0^\infty \int_0^\pi \frac{r\epsilon}{2\pi\left(r^2+\epsilon^2\right)^{\frac{3}{2}}} dr = \int_\Omega \phi_\epsilon^{[6]}(r)drdr \\ &= \pi\epsilon \int_0^\infty \frac{3r\epsilon^3}{2\pi\left(r^2+\epsilon^2\right)^{\frac{5}{2}}} = 1,\end{aligned} \tag{2.5}$$

where $\phi_\epsilon^{[10]}(r)$ is a cut-off function or blob presented by Cortez in [10]:

$$\phi_\epsilon^{[10]}(r) = \frac{3\epsilon^3}{2\pi\left(r^2+\epsilon^2\right)^{\frac{5}{2}}}. \tag{2.6}$$

According to the continuity equation, the divergence of the velocity equals to 0, therefore:

$$div\left(grad\left(P_\epsilon(r-\xi)\right)\right) = \Delta P_\epsilon(r-\xi) = \nabla\cdot\phi_\epsilon(r-\xi). \tag{2.7}$$

The corresponding regularized Green's function can be found from the homogeneous equation solution, which yields:

$$\nabla^2 G_\epsilon = \phi_\epsilon(r-\xi), \tag{2.8}$$

which by taking the Laplacian in 2-D cylindrical coordinates for radially-symmetrical functions, gives:

$$\frac{1}{r}\left[rG_\epsilon'\right]' = \frac{\epsilon}{2\pi\left(|r|^2+\epsilon^2\right)^{\frac{3}{2}}}. \tag{2.9}$$

The solution of the above equation results in the following Green's function solution kernel:

$$G_\epsilon(r) = \frac{1}{2\pi}\ln\left(\sqrt{|r|^2+\epsilon^2}+\epsilon\right), \tag{2.10}$$

Now, introducing the expression for modified radius-vector, we obtain:

$$G_\epsilon(\varrho) = \frac{1}{2\pi}\ln\varrho. \tag{2.11}$$