Engineering Analysis with Boundary Elements 000 (2017) 1–11

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

RBF-FD schemes for option valuation under models with price-dependent and stochastic volatility

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A R T I C L E I N F O

MSC:
91G20
65M70
62P05

Keywords:
Localized radial basis functions
Multiquadrics
Stochastic volatility
Constant elasticity of variance
American options

A B S T R A C T

Radial basis functions based finite difference schemes for the solution of partial differential equations have the advantage that an optimal choice of the shape parameter can yield better accuracies than standard finite difference discretisations based on the same number of nodal points. Such schemes known as local radial basis functions methods are considered for the pricing of options under the constant elasticity of variance and the Heston stochastic volatility model. A general methodology for approximating first and second order derivative terms in the finance pdes is presented and the resulting schemes are applied for option valuation. For one-dimensional problems, we derive a compact-RBF scheme which achieves a higher order accuracy when combined with a local mesh refinement strategy. Numerical results and comparisons made for European, American and barrier options illustrate the good performances of the localized radial basis functions methods.

1. Introduction

The numerical solution of partial differential equations arising in various application areas using radial basis functions (RBFs) interpolation has now gained much popularity. An RBF scheme for option pricing was introduced by Hon and Mao [1] and subsequently, several computational procedures based on RBFs have been used as an alternative to finite difference techniques for the valuation of financial derivatives. Under the dynamics of the underlying asset described by the Black–Scholes model, pricing using RBFs have been developed in [2–5] and under jump-diffusion models, RBF-based numerical pricing methodologies have been proposed in [6–9]. RBFs have also been employed for the solution of multi-asset option pricing in [10,11] and for problems with stochastic volatility in [12].

The methods mentioned above have all employed global RBFs where the differential quadrature of the derivative terms in the pricing pde is based on all the nodes inside the computational domain. Such RBFs yield spectral accuracy for smooth enough problems. However, there is a trade-off between the accuracy of the RBF collocation and the conditioning of the RBF matrix in the sense that higher accuracy comes at the price of a large condition number. Moreover the payoffs of most financial derivatives have limited continuity such that global RBF are not appropriate since dense linear systems have to be solved and spectral accuracy is not attained.

In order to avoid numerical instabilities associated with the inversion of full matrices in global RBF methods, there is increasing interest in the use of local RBF methods in which, similar to finite difference methods, the derivatives in the pde are approximated by linear combinations of local solution values. The weights in the differential quadrature are determined by requiring that the derivative be exact for the set of radial basis functions whose centres are the nodes constituting the local stencil. The local method termed as RBF-FD leads to sparse system matrices and savings in computational times.

Numerical methods based on RBF-FD for the valuation of financial contracts have been considered in [4,13,14]. The pricing of Asian options under the Black–Scholes model [15] using RBF-FD schemes and the influence of the shape parameter on price accuracy has been studied in [4]. Evaluation of the performances of local and global methods for multi-asset option pricing models and various time stepping strategies is described in [14] and it was shown that the RBF-FD matrices are well-conditioned and numerical solutions have sufficient accuracy.

Recent work have mostly focused on option pricing when the underlying asset has a log-normal diffusion. This work extends the localized RBF method to the constant elasticity of variance and stochastic volatility models. For the CEV process, a compact-RBF-FD method combined with a local mesh refinement strategy [16] is shown to yield a fourth-order accuracy.

Based on the procedure described in [17], a general methodology for approximating the weights in local RBF differential quadrature of

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https://doi.org/10.1016/j.enganabound.2017.11.003
Received 13 July 2017; Received in revised form 30 October 2017; Accepted 6 November 2017
Available online xxx
0955-7997/

the first and second derivatives are derived. These are used to develop numerical schemes for pricing options and the dependence of the root mean square error on the ratio of the shape parameter to nodal distance is studied.

This paper is structured as follows. In Section 2, the pricing problems in one and two dimensions are described and in Section 3, RBF-FD approximations for the spatial derivative terms for the one-dimensional problem are described. An improved fourth-order compact RBF scheme is presented in Section 4. The methods for the 3-point RBF-FD approximation for the one-dimensional problem are then generalized for the Heston model in Section 5. Time integration schemes for the systems of ordinary differential equations arising in the different pricing problems are described in Section 6. An extensive set of numerical tests are given in Section 7 in order to illustrate the good performances of the local RBF approach and our conclusions are presented in Section 8.

2. Option pricing models

Consider a European option on a risky asset whose payoff at maturity date T is a function of the asset price S which follows the diffusion process given by

$$dS_t = \mu S_t dt + \sigma\left(S_t\right) S_t dW_t, \quad t \ge 0, \quad S \ge 0. \tag{1}$$

The constant $\mu = r - q$ is the drift rate of the process with r being the risk-free interest rate and q, the dividend yield. For the constant volatility Black–Scholes model, $\sigma(S) = \sigma$ and the case $\sigma(S_t) = \sigma_{\text{cev}} S_t^{\beta-1}$ corresponds to the constant elasticity of variance (CEV) model where β is known as the elasticity factor. The initial instantaneous volatility at time $t = 0$ is given by $\sigma_0 = \sigma(S_0) = \sigma_{\text{cev}} S_0^{\beta-1}$. The case $\beta = 1$ corresponds to the Black–Scholes model.

Let $\tau = T - t$ denote the time to maturity of the option. Then letting $a(S) = \frac{1}{2}\sigma^2(S)S^2$ and $b(S) = \mu S$, the price $V(S, \tau)$ of the option under the scalar diffusion (1) solves the equation

$$V_\tau = a(S)V_{SS} + b(S)V_S - rV, \quad S \ge 0, \quad 0 \le \tau \le T. \tag{2}$$

For a European put option with strike price E, the payoff is $G(S) = \max(E - S, 0)$. Thus we need to solve (2) with initial condition $V(S, 0) = G(S)$ and boundary conditions given by $V(0, \tau) = Ee^{-r\tau}$ and

$$\lim_{S\to\infty} V(S, \tau) = 0.$$

The price $V(S,t)$ of an American put with maturity date T satisfies $V(S,t) \ge G(S)$. Then denoting the differential operator by

$$\mathcal{L}V = a(S)V_{SS} + b(S)V_S - rV,$$

the American put price $V = V(S, \tau)$ solves the linear complementarity problem

$$\begin{aligned}
&\frac{\partial V}{\partial \tau} \ge \mathcal{L}V, \quad S \ge 0, \quad 0 \le \tau \le T, \\
&V(S, \tau) \ge G(S), \quad S \ge 0, \quad 0 \le \tau \le T, \\
&\left(\frac{\partial V}{\partial \tau} - \mathcal{L}V\right) \cdot (V(S, \tau) - G(S)) = 0, \quad S \ge 0, \quad 0 \le \tau \le T, \\
&V(S, 0) = G(S) \\
&\lim_{S\to+\infty} V(S, \tau) = 0.
\end{aligned} \tag{3}$$

2.1. Stochastic volatility model

The Heston model [18] assumes a correlation between the asset price S_t and the asset price volatility $\sqrt{\upsilon_t}$, where υ_t is the variance process. Under this model, the risk-neutral dynamics of the stock price S_t is given by

$$\begin{aligned}
dS_t &= (r - q)S_t dt + \sqrt{\upsilon_t} S_t dW_t^{(1)}, \\
d\upsilon_t &= \kappa\left(\theta - \upsilon_t\right)dt + \sigma_\upsilon \sqrt{\upsilon_t} dW_t^{(2)},
\end{aligned} \tag{4}$$

where the standard Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$ are correlated with correlation coefficient ρ. The constant σ_υ is the volatility of the volatility process, κ is the rate of mean reversion and θ is the long-run mean of υ_t.

Let $\hat{a}(S, \upsilon) = \upsilon S^2/2$, $\hat{b}(S, \upsilon) = \rho\sigma_\upsilon \upsilon S$, $\hat{c}(\upsilon) = \sigma_\upsilon^2 \upsilon/2$, $\hat{d}(S) = (r - q)S$ and $\hat{e}(\upsilon) = (\kappa(\theta - \upsilon) - \lambda_\upsilon \sigma_\upsilon \sqrt{\upsilon})$, where λ_υ denotes the market price of volatility risk. Under the stochastic volatility model given by (4), the price $\bar{V}(S, \upsilon, \tau)$ of a European put option with strike E is the solution of the pde

$$\frac{\partial \bar{V}(S, \upsilon, \tau)}{\partial \tau} = \mathcal{L}_H \bar{V}(S, \upsilon, \tau), \quad S \ge 0, \quad \upsilon \ge 0, \quad 0 \le \tau \le T, \tag{5}$$

with the differential operator $\mathcal{L}_H$ given by

$$\mathcal{L}_H \bar{V} = \hat{a}(S, \upsilon)\frac{\partial^2 \bar{V}}{\partial S^2} + \hat{b}(S, \upsilon)\frac{\partial^2 \bar{V}}{\partial S \partial \upsilon} + \hat{c}(\upsilon)\frac{\partial^2 \bar{V}}{\partial \upsilon^2} + \hat{d}(S)\frac{\partial \bar{V}}{\partial S} + \hat{e}(\upsilon)\frac{\partial \bar{V}}{\partial \upsilon} - r\bar{V}.$$

The initial condition is given by

$$\bar{V}(S, \upsilon, 0) = G_H(S, \upsilon), \tag{6}$$

where $G_H(S, \upsilon) = \max(E - S, 0)$. Different boundary conditions that have been used in the literature for the pde (5) can be found in [19]. On the plane $S = 0$, we have

$$\bar{V}(0, \upsilon, \tau) = Ee^{-r\tau}, \tag{7}$$

and the far-field boundary conditions are

$$\lim_{S\to\infty} \frac{\partial \bar{V}}{\partial S} = 0, \quad \lim_{S\to\infty} \frac{\partial \bar{V}}{\partial \upsilon} = 0. \tag{8}$$

A discussion on boundary conditions for the case $\upsilon = 0$ can be found in [20] where it is argued that $\bar{V}(S, 0, t) = 0$ regardless of the ratio $2\kappa\theta_\upsilon/\sigma_\upsilon^2$. In [21], the boundary condition on $\upsilon = 0$ is prescribed by the requirement that the option price $\bar{V}$ is the solution of the pde

$$\frac{\partial \bar{V}}{\partial \tau} = \mathcal{L}_0 \bar{V}, \quad S \ge 0, \quad 0 \le \tau \le T.$$

where

$$\mathcal{L}_0 \bar{V} = rS\frac{\partial \bar{V}}{\partial S} + \kappa\theta\frac{\partial \bar{V}}{\partial \upsilon} - r\bar{V}.$$

The value of an American put option requires the solution of the linear complementarity problem

$$\frac{\partial \bar{V}}{\partial \tau} \ge \mathcal{L}_H \bar{V}, \quad \bar{V} \ge G_H, \quad \left(\frac{\partial \bar{V}}{\partial \tau} - \mathcal{L}_H \bar{V}\right) \cdot \left(\bar{V} - G_H\right) = 0. \tag{9}$$

with initial condition given by (6), the boundary condition on $S = 0$ given by (7) and the far-field conditions (8). On the boundary $\upsilon = 0$, the price solves the linear complementarity problem

$$\frac{\partial \bar{V}}{\partial \tau} \ge \mathcal{L}_0 \bar{V}, \quad \bar{V} \ge G_H^0, \quad \left(\frac{\partial \bar{V}}{\partial \tau} - \mathcal{L}_0 \bar{V}\right) \cdot \left(\bar{V} - G_H^0\right) = 0.$$

where $G_H^0(S) = G_H(S, 0, \tau)$.

Various approaches have been employed to solve the system of equations arising from finite difference approximations of the linear complementarity problem (9). The different approaches include a multigrid method by Clarke and Parrot [22], a penalty method by Zvan, Forsyth and Vetzal [21] and an operator splitting method by Ikonen and Toivanen [23]. Requirements on computational resources for the three methods are discussed by Zhu and Chen [20]. Ikonen and Toivanen [24] further proposed componentwise splitting methods for solving the linear complementarity problem by decomposing the discretized problem into three linear complementarity problems with tridiagonal matrices. Efficiency comparisons carried out by Ikonen and Toivanen [25] indicate that a componentwise splitting method is faster than operator splitting and penalty methods. More recent work include a finite element method with quadratic basis functions in [26] and a numerical method using Gaussian radial basis functions by Ballestra and Pacelli [12]. A pseudospectral method for a stochastic volatility model augmented with jumps is described by Ballestra and Cecere [27].

This work shows that a RBF-FD method for solving the American option problem under stochastic volatility produces accurate solutions when combined with the operator splitting method of [23].