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A novel Green element method based on two sets of nodes

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ABSTRACT

This paper presented a novel Green element method (GEM) based on two sets of nodes, one of which contains all vertices of the polygon representing the value of the pressure, called pressure nodes, the other contains all midpoints of the edges of the polygon representing the value of the normal flux, called flux nodes.

The novel method considers both the pressure and the normal flux explicitly, and utilizes the physical fact that, the algebraic sum of normal fluxes at the flux nodes which are on the edge shared by adjacent elements is zero. Therefore, the normal flux term at each internal flux node can be offset in the final global equations, and the order of the global coefficient matrix is only the number of pressure nodes. Compared to previous GEMs, the novel method has actual second-order accuracy. Moreover, because we selected the flux node at the midpoints of edges, the novel method made the solution process more in accordance with physical meaning, and the numerical solution has a better continuity in the computational domain than previous GEMs.

It is worth mentioning that, to the authors' knowledge, this is the first time to utilize two sets of nodes when estimating the numerical solution to the problem described by only one differential equation. Therefore, the novel method may bring new ideas to other numerical methods, such as finite difference or finite element method, etc.

1. Introduction

Green element method (GEM) is a promising technique to solve many engineering problems described by nonlinear differential equation. Identical to boundary element method (BEM), GEM is also based on the boundary integral equation. However, for GEM, the computational domain is divided into a finite number of elements, and the boundary integral equation is applied to each element, which is the essential difference between GEM and BEM. Moreover, GEM looks similar to finite element method (FEM), because both methods are based on a finite number of elements.

The original Green element method was proposed by Taigbenu [1-5], in which the computational domain was divided by the polygons, and the vertices of polygons were used as the solution nodes. In this method, the normal flux at each internal node was approximated by differentiating the pressure, which was evaluated by nodal values of the pressure and base functions. Therefore, this process led to a reduction of the overall accuracy. Archer [7-8] made attempts at reducing the effect of the approximation of the normal flux in the original GEM by using overhauser interpolation functions. With the process, the accuracy was improved. Archer implemented this approach only on rectangular grids and highlighted the problems of using it when the source node is on an external boundary. And also to improve the accuracy, Pecher [9] and Lorinczi [10-13] proposed a Flux-Vector-Based Green Element Method, then applied the approach to some problems in rectangular and triangular grids. The method made the flux and pressure simultaneously solved in a system of linear equations, and improved the accuracy to a second-order. The values of the pressure and flux, however, are discontinuous at the same nodes and boundary, which was not consistent with the real physical process and may bring some reduction to accuracy. Taigbenu [6] returned to the original GEM and made the nodal pressure vary as a second order degree polynomial of the spatial variables such that the approximation of the normal flux was a first order polynomial of the spatial variables, and this effort improved the accuracy. To further improve the accuracy and make the solution process more in accordance with physical significance, we proposed a novel GEM based on two sets of nodes, one of which contains all vertices of the polygons representing the value of the pressure, called pressure nodes, the other contains all midpoints of the edges of the polygon representing the value of the normal flux, called flux nodes. Setting up these two sets of nodes is mainly to deal with the normal flux term better, which can further develop GEMs. In addition, utilizing two sets of nodes when estimating numerical solution to the problem described by only one differential equation may bring other numerical methods some new ideas.

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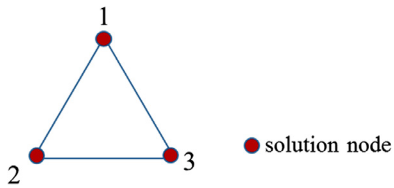

Fig. 1. Definition sketch of the triangle element for the original GEM.

2. Original Green element method

To make the numerical solution of the transient nonlinear flow in unsaturated media, Taigbenu [1] proposed GEM. The method applied the boundary integral theory to every element, which segmented the computational domain. The advantages of the method are that complex flow pattern such as transient flow in heterogeneous media can be solved well, and the global coefficient matrix is banded compared to boundary element method.

Using our notation, let us introduce main solution procedure of the original GEM. The transient flow in heterogeneous media can be described by the differential equation:

$$\nabla\cdot(K\nabla p)=c\frac{\partial p}{\partial t}+f \tag{1}$$

where p is the pressure over the computational domain, f is the distribution of internal source strengths, K and c represent the properties of media, which is a function of spatial location in heterogeneous media.

The boundary conditions along the domain may be:

(i) Dirichlet condition: $p=p_b$
(ii) Neumann condition: $\partial p/\partial\mathbf{n}=q_b$
(iii) Mixed boundary condition: $ap+b\partial p/\partial\mathbf{n}=0$

Eq. (1) is rewritten as

$$\nabla^2 p=-\nabla\psi\cdot\nabla p+\sigma\frac{\partial p}{\partial t}+\nu f \tag{2}$$

where $\psi=\ln K$, $\nu=1/K$, $\sigma=c\nu$

Define the fundamental solution (Green function) $G(M, M_i)$ in infinite space:

$$\nabla^2 G(M,M_i)=\delta(M,M_i) \tag{3}$$

where M_i is a source point in the domain, M is any point in the domain, $\delta(M, M_i)$ is the Dirac delta function.

Utilize Green second theorem with $G(M, M_i)$ provides the boundary integral equation:

$$-\lambda p_i+\int_\Gamma\left(p\nabla G\cdot\mathbf{n}+G\frac{q}{K}\right)\mathrm{d}s+\iint_\Lambda G\left[-\nabla\psi\cdot\nabla p+\sigma\frac{\partial p}{\partial t}+\nu f\right]\mathrm{d}A=0 \tag{4}$$

where $q=-K\nabla p\cdot\mathbf{n}$ is the normal flux along the boundary, subscript i denotes the selected point, λ denotes feature angle reflecting the position of the selected point in the domain.

It can be seen that, in addition to boundary integral, there is also a domain integral which is pressure related. I_Γ and I_Λ are used to denote the boundary integral and the domain integral, respectively.

To evaluate I_Γ and I_Λ, the computational domain is divided into many elements, each of which consists of N_e solution nodes at the vertices of the element. As is shown in Fig. 1, there are only one set of nodes which contains three solution nodes in the triangular element. Moreover, there are two unknown values including the pressure and the normal flux in each node. Therefore, each node has two degrees of freedom.

It is worth emphasizing that, the essence of this method is to apply the boundary integral equation to each element with corresponding I_{Γ_e} and I_{Λ_e}. In each element, similar to the finite element method, pressure p and normal flux are evaluated by means of discrete nodal values and suitable base functions ϕ_i, $i=1, 2, \ldots, N_e$:

$$p=\sum_{i=1}^{N_e}p_i\phi_i \tag{5}$$

$$q=\sum_{i=1}^{N_e}q_i\phi_i \tag{6}$$

Substitute Eqs. (5) and (6) into the boundary integral equation to evaluate I_{Γ_e} and I_{Λ_e}, as a result, it gives a system of linear equations within the element. However, each internal node has two unknown quantities, including the pressure and the normal flux, so the system of linear equations is unclosed. Taigbenu proposed a method, which evaluated the normal flux by differentiating the pressure:

$$q_n=-K\sum_{i=1}^{N_e}\frac{\partial\phi_i}{\partial n}P_i \tag{7}$$

In each element, when one pressure node i is selected, the boundary integral equation made discrete on each element can be written as:

$$\sum_{j=1}^{3}R_{ij}p_j+\sum_{z=a}^{c}L_{ij}p_j-\sum_{j=1}^{3}\sum_{m=1}^{3}U_{imj}\psi_m p_j+\sum_{j=1}^{3}\sum_{m=1}^{3}W_{imj}\left[\sigma_m\frac{\mathrm{d}p_j}{\mathrm{d}t}+\upsilon_m f_j\right]=0 \tag{8}$$

where $R_{ij}=\int_{\Gamma_e}\phi_j\nabla G(M,M_i)\cdot\mathbf{n}\mathrm{d}s-\delta_{ij}\lambda$

$$L_{ij}=\int_{\Gamma_e}G(M,M_i)\frac{\partial\phi_j}{\partial n}\mathrm{d}s$$

$$U_{imj}=\iint_{\Lambda_e}G(M,M_i)\frac{\partial\phi_m}{\partial x}\frac{\partial\phi_j}{\partial x}\mathrm{d}A+G(M,M_i)\frac{\partial\phi_m}{\partial y}\frac{\partial\phi_j}{\partial y}\mathrm{d}A$$

$$W_{imj}=\iint_{\Lambda_e}G(M,M_i)\phi_m\phi_j\mathrm{d}A$$

Thus the system of linear equations within the element can be obtained:

$$(E_{ij})_{3\times3}p_{(3)}+(C_{ij})_{3\times3}\frac{\mathrm{d}p_j}{\mathrm{d}t}+(F_i)_3=0 \tag{9}$$

In which

$$E_{ij}=\sum_{m=1}^{3}\left(R_{ij}+L_{ij}-U_{imj}\psi_m\right)$$

$$C_{ij}=\sum_{m=1}^{3}W_{imj}\sigma_m$$

$$F_i=\sum_{j=1}^{3}\sum_{m=1}^{3}W_{imj}\upsilon_m f_j$$

$$p_{(3)}=(p_1,p_2,p_3)^T$$

Substitute the time weighting factor to control the degree of quadrature and make the equation concise:

$$b=Mp^{(2)}(i=1,2,\ldots nj=1,2,\ldots n) \tag{10}$$

where $M=-(\theta E+\frac{C_{(3\times3)}}{\Delta t})$

$$b=\left((1-\theta)E-\frac{C}{\Delta t}\right)p^{(1)}+F$$