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The method of fundamental solutions for the identification of a scatterer with impedance boundary condition in interior inverse acoustic scattering

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ABSTRACT

We employ the method of fundamental solutions (MFS) for detecting a scatterer surrounding a host acoustic homogeneous medium D due to a given point source inside it. On the boundary of the unknown scatterer (assumed to be star-shaped), allowing for the normal velocity to be proportional to the excess pressure, a Robin impedance boundary condition is considered. The coupling Robin function λ may or may not be known. The additional information which is supplied in order to compensate for the lack of knowledge of the boundary ∂D of the interior scatterer D and/or the function λ is given by the measurement of the scattered field (generated by the interior point source) on a curve inside D. These measurements may be contaminated with noise so their inversion requires regularization. This is enforced by minimizing a penalised least-squares functional containing various regularization parameters to be prescribed. In the MFS, the unknown scattered field u^s is approximated with a linear combination of fundamental solutions of the Helmholtz operator with their singularities excluded from the solution domain D and this yields the discrete version of the objective functional. Physical constraints are added and the resulting constrained minimization problem is solved using the MATLAB© toolbox routine `lsqnonlin`. Numerical results are presented and discussed.

1. Introduction

Recently, the interior inverse scattering problem initiated in [6] for testing the structural integrity of a cavity has received some attention [17,18,21] due to its potential practical importance and pathway to impact proposed in [17] to model the calculation of the extent of a homogeneous reservoir from the measured data obtained from a transmitter-receiver instrument which is lowered through a borehole into the reservoir.

The numerical reconstruction of a sound-soft, i.e. perfectly conducting scatterer D on whose boundary ∂D the total field u vanishes, from measurements on an interior closed curve Γ inside D was previously investigated as follows:

– in [17], using the boundary element method (BEM) based on the single layer potential representation for the scattered field u^s and a regularized Newton minimization method;
– in [18], using the BEM based on the double layer potential representation for u^s and the linear sampling method;
– in [21], using a decomposition method based on a variant of the method of fundamental solutions (MFS), see [4,11], combined with a simple graphical method based on plotting the zero level set contours of the total field;
– in [9,10], using the MFS (or the plane waves method (PWM)) combined with a trust region reflective algorithm for minimizing the nonlinear Tikhonov regularization functional subject to constraints, implemented using the MATLAB© toolbox routine `lsqnonlin`.

Later on, the linear sampling method and the factorization method were employed in [15,19], respectively, to reconstruct a scatterer on whose boundary a homogeneous Robin boundary condition is satisfied by the total field. It is the purpose of this paper to extend the MFS analysis of [9] for the sound-soft scatterer to the more general identification of a scatterer with a Robin boundary condition which includes the sound-hard case of a perfectly insulated scatterer when $\lambda = 0$ and the sound-soft case of a perfectly conducting scatterer when $\lambda = \infty$. Moreover, the coupling function λ between the Dirichlet and Neumann data in the Robin boundary condition may be known or unknown.

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Therefore, we shall investigate three inverse problems: Problem A in which D is unknown but λ is known, problem B in which D is known but λ is unknown, and problem C in which both D and λ are unknown.

The plan of the paper is as follows. In Section 2, we present the formulations of the direct and inverse problems that are investigated. The MFS approximation of the scattered field and the numerical realization of the constrained nonlinear minimization problem are described in Section 3. Numerical results are analyzed and discussed in terms of accuracy and stability in Section 4. In particular, the influence of the regularization parameters on the stability of the reconstructions of the scatterer D and/or the Robin function λ are thoroughly investigated. Finally, conclusions and possible future work are included in Section 5.

2. Mathematical formulation

We consider the scattering with a wave number $0 < k = \omega/c$, where $c > 0$ is the speed of sound and $\omega > 0$ is the frequency of a time harmonic wave, due to a given point source $\boldsymbol{z}_0$ inside the two-dimensional, bounded and simply-connected scatterer domain D with a sufficiently smooth, e.g. C^2, [19], or Lipschitz, [15], boundary ∂D. Then the incident field is given by

$$u^{\mathrm{inc}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}, \boldsymbol{z}_0) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{z}_0|), \quad \boldsymbol{x} \in \mathbb{R}^2, \tag{2.1}$$

where i is the imaginary unit and $H_0^{(1)}$ denotes the Hankel function of first kind of order zero, and the scattered field u^{s} satisfies the Helmholtz equation

$$\Delta u^{\mathrm{s}} + k^2 u^{\mathrm{s}} = 0 \quad \text{in} \quad D. \tag{2.2}$$

Plane wave propagation in a given direction or an incoming cylindrical wave [13,14], can also be considered instead of the point source wave (2.1).

On the boundary ∂D of D we assume that a homogeneous Robin boundary condition for the total field $u = u^{\mathrm{s}} + u^{\mathrm{inc}}$ holds, namely,

$$\frac{\partial u}{\partial \nu} + \mathrm{i}\, \lambda\, u = 0, \quad \text{on} \quad \partial D, \tag{2.3}$$

where $\boldsymbol{\nu}$ is the outward unit normal to ∂D and $0 < \lambda \in C(\partial D)$ or $L^\infty(\partial D)$ is a Robin coupling real function usually called the impedance function, [15], or admittance, [8]. When $\lambda \to 0$ or $\lambda \to \infty$ we obtain the particular cases of a sound-hard or sound-soft scatterer, respectively. However, unlike these ideal cases, the Robin impedance boundary condition (2.3) with $0 < \lambda < \infty$ is more realistic because, in practice, scatterers are never perfect and the waves always penetrate a little through the boundary ∂D, with λ characterising the level of penetration.

Using (2.1) we can recast (2.3) as a non-homogeneous Robin boundary condition for the scattered field given by

$$\begin{aligned} \frac{\partial u^{\mathrm{s}}}{\partial \nu}(\boldsymbol{x}) + \mathrm{i}\, \lambda\, u^{\mathrm{s}}(\boldsymbol{x}) = {} & \frac{k\,\mathrm{i}}{4} H_1^{(1)}(k|\boldsymbol{x} - \boldsymbol{z}_0|)\, \frac{(\boldsymbol{x} - \boldsymbol{z}_0)\cdot \boldsymbol{\nu}(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{z}_0|} \\ & + \frac{\lambda}{4} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{z}_0|), \quad \boldsymbol{x} \in \partial D, \end{aligned} \tag{2.4}$$

where $H_1^{(1)}$ is the Hankel function of the first kind of order one.

2.1. The direct problem

When D and λ are known, equations (2.2) and (2.4) form the direct problem which is well-posed in $C^2(D) \cap C^1(\overline{D})$ or $H^1(D)$, [2,3]. In the case of the Dirichlet ($\lambda \to 0$) or Neumann ($\lambda \to \infty$) boundary conditions we need to add the assumption that k^2 is not a Dirichlet or Neumann eigenvalue of $-\Delta$ in D, respectively.

2.2. The inverse problems

We consider the following inverse problems under the general assumption that:

(α) k^2 is not a Dirichlet eigenvalue for $-\Delta$ in the interior Ω of the curve Γ introduced below. Note however that this assumption is not so essential as we can always rescale Γ, [21].

2.2.1. Inverse problem A

Solve the Helmholtz equation (2.2) for the scattered field u^{s} subject to the Robin boundary condition (2.4) with given λ but unknown boundary ∂D which also has to be determined from additional measurements of u^{s} on some known interior closed curve Γ assumed to lie inside D. The condition that $\boldsymbol{z}_0 \in \Gamma$ is not essential but in the sequel we shall assume, for simplicity, that Γ is the circle of radius $|\boldsymbol{z}_0| > 0$ centred at the origin, i.e.

$$\Gamma = \partial B_{|\boldsymbol{z}_0|}(\boldsymbol{0}). \tag{2.5}$$

Then the above additional condition is

$$u^{\mathrm{s}}(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{2.6}$$

where f is some given measured data which may be contaminated with noise.

2.2.2. Inverse problem B

In this case we again consider the Helmholtz equation (2.2) for the scattered field u^{s} subject to the Robin boundary condition (2.4) but now the boundary ∂D is known and the impedance $\lambda(\boldsymbol{x})$ is unknown. The additional measurements are again given by (2.6).

2.2.3. Inverse problem C

Now both the boundary ∂D and the impedance $\lambda(\boldsymbol{x})$ in (2.4) are unknown.

At this stage, we briefly discuss the way the data f, obtained from the measurement of the scattered field Γ, could be interpreted. We first observe that the data f in expression (2.6) is rather limited because it only contains the measurement obtained from a single point source $\boldsymbol{z}_0 \in D$. Also, (2.6) can be further restricted to a limited aperture case by only specifying it on a subportion Γ_1 of Γ. We have also fixed the wave number k. So, we can remark that in some practical applications it may be possible to measure more data obtained by varying the wave number k or the point source $\boldsymbol{z}_0$ along Γ. Thus, in general, for compatible data the function f in (2.6) depends on both $\boldsymbol{z}_0$ and k. In particular, for fixed k satisfying assumption (α), but varying $\boldsymbol{z}_0 \in \Gamma$ so that (2.6) reads as a matrix of measured data

$$u^{\mathrm{s}}(\boldsymbol{x}; \boldsymbol{z}_0) = f(\boldsymbol{x}; \boldsymbol{z}_0), \quad \boldsymbol{x}, \boldsymbol{z}_0 \in \Gamma, \tag{2.7}$$

then a solution of inverse problem C given by (2.2), (2.4) and (2.7) is unique, [19]. However, this uniqueness result requires the measurement $u^{\mathrm{s}}(\cdot\,; \boldsymbol{z}_0)$ for infinitely many point sources $\boldsymbol{z}_0 \in \Gamma$ which may become impractical. We note that for a single fixed source $\boldsymbol{z}_0 \in \Gamma$, the uniqueness of the restricted inverse problem A given by (2.2), (2.4) and (2.6) is only known when we assume *a priori* that D is a disk, [15], a small and smooth perturbation of a disk, [12], or in the sound-soft case ($\lambda \to \infty$), by requiring D to be contained in a disk of radius t_0/k, where $t_0 = 2.40482$ is the smallest positive zero of the Bessel function J_0.

On first solving the direct well-posed Dirichlet problem for the Helmholtz equation given by (2.2) in Ω and (2.6), with assumption (α), the normal derivative

$$\frac{\partial u^{\mathrm{s}}}{\partial \nu}(\boldsymbol{x}) =: g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{2.8}$$

can be obtained. It then means that (2.6) and (2.8) are compatible Cauchy data for the Helmholtz equation (2.2) in the annular domain $D \backslash \Omega$. From the unique continuation property of the Helmholtz equation it follows that the Cauchy data u^{s} and $\partial_\nu u^{\mathrm{s}}$ on ∂D are uniquely determined. Then, in principle, provided that $u \neq 0$ almost everywhere on ∂D the coefficient λ could be determined directly from (2.3) as

$$\lambda(\boldsymbol{x}) = \mathrm{i}\, \frac{\partial_\nu u(\boldsymbol{x})}{u(\boldsymbol{x})}, \quad \boldsymbol{x} \in \partial D. \tag{2.9}$$

However, this direct method was found to be less accurate and stable than a regularized nonlinear least-squares method based on approximating λ with a finite linear combination of trigonometric functions, [19].