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An upwind local radial basis functions-differential quadrature (RBF-DQ) method with proper orthogonal decomposition (POD) approach for solving compressible Euler equation

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A B S T R A C T

The current paper is an improvement of the developed technique in Shu et al. (2005). The proposed improvement is to reduce the used CPU time for employing the local radial basis functions-differential quadrature (LRBF-DQ) method. To this end, the proper orthogonal decomposition technique has been combined with the LRBF-DQ technique. For checking the ability of the new procedure, the compressible Euler equation is solved. This equation has been classified in category of system of advection–diffusion equations. Moreover, several test problems are given that show the acceptable accuracy and efficiency of the proposed scheme.

1. Introduction

During the recent decade, the meshless methods have been employed to solve the PDEs. The meshless methods do not use any mesh, element or lattice to discrete the computational domain for obtaining some numerical results. According to the basic advantages of meshless methods, these techniques may be classified as follows:

- The global form,
- The local form.

The local meshless method is an improvement of meshless techniques where they can be split in two forms:

- Local meshless methods based on the variational (local) weak form,
- Local meshless methods based on the strong form.

In the local meshless methods based on the weak form, there are some integrals which must be computed with suitable accuracy thus these methods have more difficulty and need more CPU time. But in the local meshless methods based on the strong form there are not any integral so these techniques will be very flexible to solve models with nonlinear term.

A local meshless collocation method based on the finite difference approach is the RBFs finite difference (RBFs-FD) method. The RBF-FD idea has been developed in [21,28,29,53,58,62,63]. Authors of [27] developed a filter approach for RBF-FD that is related to traditional hyperviscosity and which can be applied quickly in any number of dimensions. Also, some analytical explanations related to the weights of Gaussian RBF-FD formula are obtained in [7]. The main aim of [5,6,8] is to obtain an optimal shape parameter for RBF-FD technique. Also, some researchers studied RBF-FD method such as large-scale geoscience modeling [26], hyperbolic PDEs on the sphere [10], diffusion and reaction-diffusion equations (PDEs) on closed surfaces [55], multi-dimensional Cahn–Hilliard, Swift–Hohenberg and phase field crystal equations [16], multi-dimensional Vlasov–Poisson and Vlasov–Poisson–Fokker–Planck systems arising in plasma physics [17], etc. Also, the RBF approach is applied on financial mathematics such as an increasingly popular and promising approach to solve option pricing models [38,39], a numerical method to compute the survival (first-passage) probability density function in jump-diffusion models [3], the survival (first-passage) probability density function of jump-diffusion models with two stochastic factors [4], etc. Also, authors of [2] developed a new RBF method in which the inversion of large system matrices is ignored. They proposed this technique by combining Gaussian radial basis functions with a suitable operator splitting scheme [2].

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One of the local meshless methods is the RBF-differential quadrature (RBF-DQ) procedure. The differential quadrature method was first introduced by Bellman et al. [9]. The polynomial functions have been selected as the test function [56]. For the first time, authors of [58] proposed the meshless RBF-DQ method by using the RBFs. The RBF-DQ method is similar to the LRBF and RBF-FD methods. The RBFs-DQ is employed for solving several PDEs such as equations in fluid dynamic [58,59], system of boundary value problems [18], coupled Klein–Gordon–Zakharov equations [19], one- two- and three-dimensional Cahn–Hilliard (CH) equations [20], doubly-curved shells made of composite materials [64], Stokes flow problem in a circular cavity [41], etc. Natural phenomena can be described by partial differential equations (PDEs). We refer the interested reader to [67] for various applications of partial differential equations in science and engineering and also for some approaches in obtaining their solutions.

The main aim of the current paper is to develop a combined local RBF-DQ approach to solve compressible Euler equation. The local RBF-DQ approach is constructed by combining radial basis functions concept and differential quadrature method. In the finite local differential quadrature, the corresponding weights can be obtained by using local polynomial approximations. Also, radial basis functions can be chosen instead as basis functions translates of radially symmetric functions. Thus, combination of radial basis functions with local differential quadrature approach leads to radial basis function-generated DQ formulas. Furthermore, all approximations again local, but nodes can now be placed freely. Also, to reduce the used CPU time in the local RBF-DQ technique, we combined the local RBF-DQ with the proper orthogonal decomposition (POD) method.

1.1. Organization chart for the manuscript

In this paper, we apply a local truly meshless method based on the RBF-DQ technique for solving an equation in water science in two-dimensional case.

The structure of this article is as follows:

- *In* Section 2, *we explain the LRBF-DQ.*
- *In* Section 5, *we explain the snapshot collection and POD basis.*
- *In* Section 6, *we report the numerical experiments of solving the considered models for some test problems.*
- *Finally, a brief conclusion of the current paper has been written in* Section 7.

2. The local RBFs-DQ method

The local RBFs-DQ method approximates the unknown function using the RBFs and it estimates mth derivative via differential quadrature (DQ) technique. Let $\Omega \subset \mathbb{R}$ and $x_i \in \Omega$ be arbitrary in which this point has a support domain with n_i nodes $\{x_0^i, x_1^i, \ldots, x_{n_i}^i\} \subseteq \{x_1, x_2, \ldots, x_N\}$ that $n_i < N$ inside its support domain as is described in Fig. 1. The main aim of the DQ method is approximating the mth derivative at a reference point by a smooth function as follows

$$\left.\frac{d^m u(x)}{dx^m}\right|_{x=x_i} = \sum_{j=0}^{n_i} w_{i,j}^{m,x} u(x_j^i), \quad i = 0, 1, 2, \ldots, N. \tag{2.1}$$

In the local RBFs-DQ, the RBFs can be selected as a set of base functions. Then the mth derivative of unknown function $u(x)$ at point x_i is approximated as

$$\left.\frac{d^m \varphi_j}{dx^m}\right|_{x=x_i} = \sum_{j=0}^{n_i} w_{i,j}^{m,x} \varphi_j(x_j^i), \qquad j = 0, 1, 2, \ldots, n_i, \tag{2.2}$$

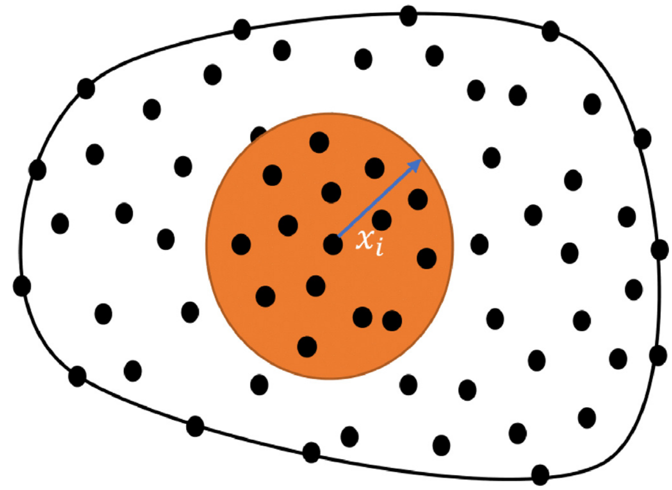

Fig. 1. An arbitrary computational domain with a center point and its support domain.

in which φ_j is a radial basis function. Let coefficient matrix $[A]$ be non-singular, then by solving the following system

$$\underbrace{\begin{bmatrix} \frac{d^m\varphi_0(x_i)}{dx^m} \\ \frac{d^m\varphi_1(x_i)}{dx^m} \\ \vdots \\ \frac{d^m\varphi_{n_i}(x_i)}{dx^m} \end{bmatrix}}_{\left[\frac{d^m\varphi(x_i)}{dx^m}\right]} = \underbrace{\begin{bmatrix} \varphi_0(x_0^i) & \varphi_0(x_1^i) & \ldots & \varphi_0(x_{n_i}^i) \\ \varphi_1(x_0^i) & \varphi_1(x_1^i) & \ldots & \varphi_1(x_{n_i}^i) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n_i}(x_0^i) & \varphi_{n_i}(x_1^i) & \ldots & \varphi_{n_i}(x_{n_i}^i) \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} w_{i,0}^{(m)} \\ w_{i,1}^{(m)} \\ \vdots \\ w_{i,n_i}^{(m)} \end{bmatrix}}_{[w]}, \tag{2.3}$$

we obtain

$$[w] = [A]^{-1}\left[\frac{d^m\varphi(x)}{dx^m}\right]. \tag{2.4}$$

Definition 2.1. [25,68] A real valued continuous function $\phi \in \mathbb{R}^d \longrightarrow \mathbb{C}$ is positive definite if for all sets $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ of distinct points and all vectors $\lambda \in \mathbb{R}^d$

$$\lambda^T \phi \lambda = \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \phi(x_i - x_j) \geq 0. \tag{2.5}$$

Also, the function ϕ is called strictly positive definite on $\mathbb{R}^d$ if the quadratic form (2.5) is zero only for $\lambda = 0$.

We interpolate a continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ on a set $X = \{x_1, \ldots, x_N\}$ with choosing the radial basis function for $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ that is radial in the sense that $\phi(x) = \Psi(\|x\|)$, where $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^d$ as we will explain it in the next section. Now, we assume ϕ to be strictly positive definite, then the interpolation function has the following form [25,68]

$$\mathcal{I}(f(x)) = \sum_{i=1}^{N} \lambda_i \phi(x - x_i). \tag{2.6}$$

The basic problem is to find N unknown coefficients λ_i in which N interpolation conditions are to the following form [25,68]

$$\mathcal{I}(f(x_i)) = f_i, \quad i = 1, \ldots, N. \tag{2.7}$$

It has proved that the interpolation matrix based on a strictly positive definite function is nonsingular [25,68]. In the following, we mention some strictly positive definite functions:

1. Gaussian:

$$\phi(r) = \exp\left(-(cr)^2\right), \tag{2.8}$$

2. Linear generalized IMQ:

$$\phi(r) = \frac{2 - (cr)^2}{\left(1 + (cr)^2\right)^4}. \tag{2.9}$$