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On the application of the method of fundamental solutions to nonlinear partial differential equations

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A B S T R A C T

This paper addresses the application of a domain-type method of fundamental solutions (MFS-D) together with a Picard iteration scheme for solving nonlinear elliptic partial differential equations. A mathematical justification of the iterative process and an *a posteriori* error estimate are provided for a class of nonlinear problems. Numerical simulations illustrate the convergence and accuracy of the method, in several examples, including a case for a sine-Gordon equation.

1. Introduction

The method of fundamental solutions (MFS) is a meshfree method for solving partial differential equations (PDEs). Being based on point sources located outside the domain where the solution is sought and satisfying the underlying linear homogeneous PDE, the MFS is considered a Trefftz type method. Its mathematical justification was addressed for the first time more than fifty years ago by Kupradze and Alekside [15], and among the first computational implementations we find the work of Oliveira [18]. In the following two decades, the MFS was readdressed in several reference papers, including, for instance, those by Mathon and Johnston [17], devoted to the choice of basis functions, and Bogomolny [5], treating the convergence analysis. The review articles by Fairweather and Karageorghis [9] or Golberg and Chen [11] contain a detailed introduction to the MFS.

In the context of meshfree methods for partial differential equations, the MFS regained attention in the last decades, both in the applied mathematics and engineering communities. Its main advantage, as compared with finite differences and finite elements methods, is that it offers higher accuracy, under appropriate regularity assumptions, even though there is an extra cost due to the ill-conditioning nature of the resulting linear systems [6,7,12].

The original MFS was restricted to linear homogeneous boundary value problems described by $Du = 0$, in a domain $\Omega \subset \mathbb{R}^d$, and $\mathcal{B}u = g$, on $\partial\Omega$, where D denotes a (generic) elliptic differential operator with fundamental solution, and $\mathcal{B}$ denotes a linear boundary operator. Several ways of extending the method to the case of nonhomogeneous source/force terms $Du = f$ have been developed in the last years. Such methods are mostly based on the construction of a particular solution u_P, satisfying $Du_P = f$, and on solving, by the MFS, the homogeneous problem $Du_H = 0$ with the boundary condition $\mathcal{B}u_H = g - \mathcal{B}u_P$. Then, the solution of the original problem is just given by $u = u_P + u_H$. In order to obtain u_P, the so-called method of particular solutions can be applied, which consists in interpolation techniques, using radial basis functions (RBF), polynomials, splines, and other basis functions.

The MFS has been applied to nonlinear problems together with the RBF method of particular solutions and Picard iteration in several papers, *e.g.* [13,14,19,21,23]. Other possibilities to overcome some specific nonlinearities in functionally graded materials, include the use of the Kirchoff transformation to reduce the problem to a linear one, as done in [16] or in [10]. Another approach to handle the nonlinear terms is the use of the homotopy analysis method (HAM), instead of the Picard iteration scheme, as in [20,22].

The MFS-D, a domain-type method of fundamental solutions, was proposed in [3] (see also [1,2]) for Poisson problems and consists in the use of Helmholtz fundamental solutions as basis functions. This idea was further implemented in [4] to avoid the separation in the homogeneous and particular solutions parts. However the application of the MFS-D to solve nonlinear PDEs was not addressed in previous publications.

In this paper, we describe the main steps of an ongoing project devoted to the application of the MFS-D to nonlinear PDEs where $D = \Delta$ is the relevant linear operator. Here only the case of a nonlinearity of the form $\mathcal{A} = \mathcal{A}(u)$ is considered. In future works, the Picard iteration scheme will be tested to handle nonlinearities incorporating ∇u as well. In Section 2, we provide a justification of the iterative level of the method by means of the Banach fixed point Theorem. In Section 3, we describe the implementation of MFS-D for each linear problem of the iterative process. We also propose a control of the error in the iterative process through *a posteriori* residual estimates, allowing a stopping criteria that is independent of the approximation restrictions forced by the MFS-D discretization. This is explained in Section 4. In order to illustrate the fast convergence, accuracy and small computational effort of

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the described method, we consider, in Section 5, two examples of nonlinear elliptic equations for which we present numerical simulations in 2D.

2. A class of nonlinear elliptic PDEs

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with locally Lipschitz boundary $\Gamma = \partial\Omega$. Consider a second order nonlinear elliptic Dirichlet problem written in the form

$$\begin{cases} \Delta u - \mathcal{A}(u) = f, \text{ in } \Omega \\ u - \mathcal{B}(u) = g, \text{ on } \Gamma \end{cases} \tag{2.1}$$

where the operator $\mathcal{A}$ describes the nonlinear part of the differential equation, and $\mathcal{B}$ describes possible nonlinear boundary conditions. When $\mathcal{A} \equiv 0, \mathcal{B} \equiv 0$, we recover the classical Poisson equation with source term f and Dirichlet boundary condition g. We assume the existence of $\tilde{g} \in H^1(\Omega)$ such that its trace on Γ is equal to g, $u|_\Gamma = g$, and conditions on $\mathcal{A}$ and $\mathcal{B}$ that ensure that the problem (2.1) has an unique solution in appropriate functional spaces. To simplify the presentation, we will consider just the Dirichlet problem with $\mathcal{B} \equiv 0$. Extensions to other type of boundary conditions, including nonlinearities, might be considered.

2.1. Picard iteration

In order to apply the MFS-D, we consider a Picard iteration scheme which leads to the resolution of a linear Poisson problem in each iteration step. More precisely, given an approximation u_k we find the new iteration u_{k+1} by solving the following Poisson problem

$$\begin{cases} \Delta u_{k+1} = F_k := \mathcal{A}(u_k) + f, \text{ in } \Omega, \\ u_{k+1} = g, \text{ on } \Gamma. \end{cases} \tag{2.2}$$

Under a smallness condition involving the data, this iteration procedure may converge to an unique solution u of problem (2.1) when $k \to \infty$, and this justifies the application of this fixed point method.

The reduction to Poisson problems in each iteration is a way to convert the resolution of a nonlinear problem into a sequence of simpler linear problems. However, it should be noticed that also Helmholtz linear problems could be considered in the iteration process.

2.2. Contractivity and the fixed point theorem

At each iteration step k we have to solve a Poisson problem

$$\begin{cases} \Delta u = F_k, \text{ in } \Omega \\ u = g, \text{ on } \Gamma. \end{cases} \tag{2.3}$$

This linear problem (2.3) is well posed for each fixed $g \in H^{1/2}(\Gamma)$, and we may introduce the operator

$$\Delta_g^{-1} : F \in H^{-1}(\Omega) \mapsto u \in H^1(\Omega) \tag{2.4}$$

where u stands for the (unique) solution of (2.3), given F in the right hand side. The well posedness means that there is a constant $M_\Omega > 0$, that only depends on the domain Ω, such that the solution u verifies the elliptic estimate (see [8]):

$$\|u\|_{H^1(\Omega)} \le M_\Omega \big(\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} \big). \tag{2.5}$$

We can write (2.2) as $u_{k+1} = \Delta_g^{-1}(F_k) = \Delta_g^{-1}(\mathcal{A}(u_k) + f) := \mathcal{G}(u_k)$, with $\mathcal{G}$ the iterated operator, defined in a subset of $\{u \in H^1(\Omega) : u|_\Gamma = g\}$.

In order to apply the Banach fixed point theorem to problem $u = \mathcal{G}(u)$, and to conclude existence and uniqueness of the solution u of (2.1) along with the convergence of the Picard iterations (2.2), specific conditions should be required for the operator $\mathcal{A}$ describing the nonlinearity and for the operator $\mathcal{G}$.

Theorem 1. *Let $\mathcal{X} \subseteq \{u \in H^1(\Omega) : u|_\Gamma = g\}$ be non empty and closed. Suppose that*

(i) $\mathcal{A} : \mathcal{X} \to H^{-1}(\Omega)$ *and there exists a constant* $L \ge 0$ *such that*

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{H^{-1}(\Omega)} \le L\|u - v\|_{H^1(\Omega)}, \ \forall u, v \in \mathcal{X}, \tag{2.6}$$

(ii) $\mathcal{G} : \mathcal{X} \to H^1(\Omega)$ *defined by* $\mathcal{G}(v) = \Delta_g^{-1}(\mathcal{A}(v) + f)$ *satisfies* $\mathcal{G}(\mathcal{X}) \subseteq \mathcal{X}$,

(iii) *the constant* M_Ω *in estimate (2.5) is such that* $K := M_\Omega L < 1$.

Then the sequence $\{u_k\}$ of Picard iterations $u_{k+1} = \mathcal{G}(u_k)$ converges to the unique solution $u \in \mathcal{X}$ of problem (2.1), for arbitrary $u_0 \in \mathcal{X}$ (in particular, when $u_0 = \tilde{g}$). Moreover, the a posteriori error estimate

$$||u - u_k||_{H^1(\Omega)} \le \frac{K}{1-K} ||u_k - u_{k-1}||_{H^1(\Omega)} \tag{2.7}$$

is valid.

Proof. Under the stated conditions, the operator $\mathcal{G}$ is a contraction in $\mathcal{X}$, because

$$\begin{aligned} ||\mathcal{G}(u) - \mathcal{G}(v)||_{H^1(\Omega)} &= ||\Delta_0^{-1}(\mathcal{A}(u) - \mathcal{A}(v))||_{H^1(\Omega)} \\ &\le M_\Omega ||\mathcal{A}(u) - \mathcal{A}(v)||_{H^{-1}(\Omega)} \\ &\le M_\Omega L ||u - v||_{H^1(\Omega)} \end{aligned} \tag{2.8}$$

and $K = M_\Omega L < 1$.

By the Banach fixed point Theorem (see, for instance, [8]), $\mathcal{G}$ being a contraction and leaving the closed space $\mathcal{X}$ invariant, there is only a solution to $u = \mathcal{G}(u)$ and the sequence $\{u_k\}$ generated by $u_{k+1} = \mathcal{G}(u_k)$ converges to u, given any $u_0 \in \mathcal{X}$. The estimate (2.7) is also a consequence of the Banach fixed point Theorem. □

Remark 2. The condition $K = M_\Omega L < 1$ that allows to prove the contractivity of the operator $\mathcal{G}$ may be rather heavy, and eventually be relaxed. As our main goal is to present the application of the method, we will not get into further details on this theoretical aspect.

Remark 3. The assumption on the initial condition $u_0 = g$ is not necessary, as long as $u_k \in \mathcal{X}$, for some $k \in \mathbb{N}_0$ and this is a consequence of solving the Poisson problem (2.3), since $u_k \in H^1(\Omega)$ and $u_k = g$ on Γ.

3. Description of the numerical method

To solve each Poisson problem (2.3), we consider the fundamental solutions Φ_λ of the Helmholtz equation, which satisfy

$$\Delta\Phi_\lambda + \lambda^2\Phi_\lambda = -\delta, \tag{3.1}$$

here δ stands for the Dirac delta distribution. In the 2D case, these fundamental solutions are given explicitly in terms of the Hänkel function (i stands for the imaginary unit)

$$\Phi_\lambda(x) = \frac{\mathrm{i}}{4} H_0^{(1)}(\lambda||x||), \tag{3.2}$$

and when $\lambda = 0$, it reduces to the fundamental solution of the Laplace equation

$$\Phi_0(x) = \frac{1}{2\pi} \log|x|. \tag{3.3}$$

Remark 4. The method can be extended to other dimensions, in particular to 3D, by using the appropriate fundamental solution $\Phi_\lambda(x) = \frac{e^{i\lambda||x||}}{4\pi||x||}$ which also holds for $\lambda = 0$.

We emphasize that, since we are dealing with a nonhomogeneous PDE, the standard MFS is not applicable. However, the MFS can be extended, by using the above fundamental solutions of the Helmholtz equation to approximate the source terms F_k. This is convenient because these fundamental solutions verify, for $x \neq 0$,

$$\Delta\Phi_\lambda(x) = -\lambda^2\Phi_\lambda(x). \tag{3.4}$$

This method was introduced in [2] and it has been called the MFS-D. The MFS-D, like the standard MFS, uses a source set $\hat{\Gamma} = \partial\hat{\Omega}$, where $\hat{\Omega} \supset \Omega$ is a larger open set. $\hat{\Gamma}$ acts as an artificial boundary and an interval of frequencies is also used to prove density of the linear combinations of $\Phi_\lambda(x - y)$ in $H^1(\Omega)$.