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The method of particular solutions with polynomial basis functions for solving axisymmetric problems

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A B S T R A C T

In this paper, we extend the previous work of Chen et al. (Numer Methods Partial Differential Eq 21: 349–367, 2005) on the two-step method of particular solutions (MPS) for solving the Poisson equation with an axisymmetric forcing term and boundary conditions in an axisymmetric geometry to general differential equations and time-dependent problems using the one-step MPS. Polynomial basis functions are sufficient for the proposed approach instead of using Chebyshev polynomials. Furthermore, no boundary method is required for solving the homogeneous equation which is required in the two-step approach. In the solution process of the two-step MPS, we only require the closed form particular solution of the Laplacian or Helmholtz equation with respect to the monomial basis functions. The proposed approach is more simplified compared to the previous work and also allows us to solve a large class of partial differential equations including those with variable coefficients. We further extend the proposed approach to time-dependent problems using the Houbolt method, which is a third order time marching finite difference scheme. In the numerical implementation, we compare the results using reduced axisymmetric equations and the original 3D equations. Numerical results show the high simplicity, accuracy, and efficiency of the proposed numerical method.

1. Introduction

During the past several decades, the traditional mesh-based methods such as the finite element method, finite difference method, and boundary element method have been fully developed for solving various types of problems in science and engineering. For higher dimensional problems, one obstacle of implementing these methods is the tedious procedure of meshing the domain and its boundary. When the geometry of the given 3D domain is axisymmetric, which is formed by rotating a 2D plane region about the z-axis, the procedure can be reduced to solving the axisymmetric version of the original equation. If the forcing term of the given differential equation and the boundary conditions are also axisymmetric, the given 3D problem reduces to a 2D problem in the radial coordinate r and the axial coordinate z. As a result, the mesh generation of the domain in 2D is much easier than its original 3D version. Various numerical techniques have been developed for solving axisymmetric problems. Karageorghis [12,13] considered homogeneous axisymmetric problems with axisymmetric and non-axisymmetric boundary conditions using the boundary method. In contrast, our proposed method can be applied for solving both homogeneous and nonhomogeneous equations. In this paper, we only consider the forcing term and boundary conditions to be both axisymmetric. Otherwise, a similar approach in [13] can be applied for a non-axisymmetric forcing term or boundary conditions. We would like to refer readers to References [14,15,20,21,24] and the references cited therein for further information of axisymmetric problems.

In recent years, meshless methods have become very popular for solving various kinds of partial differential equations without the tedious mesh generation. In this paper, we will consider applying meshless methods to further avoid the mesh generation. Meshless methods using radial basis functions (RBFs) are well known and have been successfully applied for solving various types of differential equations. However, because the axisymmetric Laplacian equation does not have constant coefficients, the standard technique of RBF collocation methods can not be easily applied. Chen et al. [1] applied the two-step method of particular solutions (MPS) using polynomials as the basis functions. The closed-form particular solutions for monomial functions had been derived in this paper. Polynomials are notorious for being unstable due to the severe ill-conditioning of the resultant matrix. As a result, Chebyshev polynomials were employed as the basis functions to evaluate the particular solutions. In the solution process, each Chebyshev polynomial was expanded as the sum of a series of monomials and the particular solution

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for each monomial term was then evaluated [25–28]. In the two-step MPS, after the particular solutions are evaluated, the given problem is reduced to a homogeneous problem. A boundary method, such as the method of fundamental solutions (MFS) [4,5,7,10,16], is required to solve the remaining homogeneous problem. As such, the fundamental solution, which is available only for certain linear differential operators, is normally required. In [1], only the Poisson equation is considered. In [21], closed-form particular solutions for Helmholtz-type equations were derived, but no numerical implementation was conducted. Tsai et al. [24] extended the results in [1] to the polyharmonic and poly-Helmholtz equations using the two-step MPS.

In [2,3,30], the one-step MPS has been proposed for solving various types of partial differential equations including those with variable coefficients using RBFs. In recent years, the new approach of the MPS has further extended for solving challenging problems [8,9,31]. However, the closed-form particular solutions for axisymmetric equations using RBFs are not available. In this paper, we propose using the one-step MPS with polynomial basis functions for solving axisymmetric problems including equations with variable coefficients. For comparison reasons, we also apply this method for solving 3D problems directly [6,17]. We then compare the numerical results of these two approaches. To alleviate the difficulty of severe ill-conditioning of polynomial functions, the multiple scale technique [18,19] is employed to reduce the condition number. Unlike the two-step MPS using the Chebyshev polynomial and the MFS, the proposed one-step MPS is simple to implement. Furthermore, the proposed method is applicable to a large class of partial differential equations.

After solving Helmholtz-type equations, we extend the proposed method to axisymmetric time-dependent problems. The Houbolt method [11,23,29], which is a third order finite difference time marching scheme, is used to transform the time-dependent problem into a series of inhomogeneous modified Helmholtz equations. Using the above mentioned two-step MPS, we can obtain the solution at each time step.

The paper is organized as follows. In Section 2, we present the formulation of the one-step MPS for axisymmetric problems. In Section 3, we give a brief review of the MPS for solving 3D problems directly. The multiple scale method is presented in Section 4 as a preconditioner to reduce the condition number of the resultant matrices due to the uses of high order polynomial basis functions. In Section 5, we present how to convert the heat equation to the modified axisymmetric Helmholtz equation by the Houbolt method. In Section 6, three numerical examples are given to show the effectiveness of the proposed approaches. Finally, we present the advantages and disadvantages of the proposed approach as well as some conclusions and ideas for future work in Section 7.

2. The MPS for axisymmetric problems

In this paper, we first consider the following modified Helmholtz equation

$$\Delta u(x,y,z) - \lambda^2 u(x,y,z) = f(x,y,z), \qquad (x,y,z) \in \Omega, \tag{1}$$

$$Bu(x,y,z) = g(x,y,z), \qquad (x,y,z) \in \partial\Omega, \tag{2}$$

where Δ is the Laplacian, B the boundary operator, λ a positive constant, and Ω a bounded and connected domain with boundary $\partial\Omega$, which we assume to be piecewise smooth. f and g are given smooth functions.

If the domain Ω, the forcing term f, and the boundary conditions g are axisymmetric, then the 3D problem (1) and (2) can be reduced to solving the following 2D axisymmetric version of the modified Helmholtz equation

$$\left(\mathcal{L} - \lambda^2\right) u(r,z) = \hat{f}(r,z), \qquad (r,z) \in D, \tag{3}$$

$$Bu(r,z) = \hat{g}(r,z), \qquad (r,z) \in \partial D, \tag{4}$$

where

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

As mentioned in the previous section, RBF collocation methods are not suitable for solving axisymmetric problems. Due to the new development of the one-step MPS [2,3,30], the solution procedure has been extended for solving more general differential equations and is also much simpler than the two-step MPS.

The key step of the MPS is the derivation of the particular solution with respect to the given differential equation and basis functions. Consider the following axisymmetric equation

$$\left(\mathcal{L} - \lambda^2 I\right)\Psi(r,z) = r^k z^m, \tag{6}$$

where $k \geq 0$ and $m \geq 0$ are integers, and I is the identity operator. In [21], the closed-form particular solution, $\Psi(r, z)$, has been derived as follows

$$\Psi(r,z) = \begin{cases} -(2s)!!^2 m! \displaystyle\sum_{i=0}^{s}\sum_{j=0}^{[B/2]} \frac{(i+j)!\, r^{2s-2i} z^{m-2j}}{(2s-2i)!!^2 (m-2j)!\, i!\, j!\, \lambda^{i+j+1}}, & k = 2s, \\ (2s+1)!!^2 m! \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{[B/2]} \frac{(-1)(i+j)!\, \lambda^i r^{2s+2i+2j+3} z^{m-2j}}{(2s+2i+2j+3)!!^2 (m-2j)!\, i!\, j!}, & k = 2s+1, \end{cases} \tag{7}$$

where $[x]$ is the largest integer less than or equal to x, and $0!! = 1$, $1!! = 1$, $2!! = 2$, and

$$i!! = \begin{cases} 2\cdot 4\cdot 6 \ldots i, & \text{if } i \text{ is an even positive integer, } (i > 2), \\ 1\cdot 3\cdot 5 \ldots i, & \text{if } i \text{ is an odd positive integer, } (i > 1). \end{cases}$$

Note that $\Psi(r, z)$ in (7) is too tedious for the numerical implementation. Instead, as shown in [1], we can consider the following closed-form particular solution of the axisymmetric Poisson equation

$$\mathcal{L}\Phi(r,z) = r^k z^m, \tag{8}$$

where

$$\Phi(r,z) = \sum_{l=0}^{[m/2]} \frac{(-1)^l m!}{(m-2l)!} \left(\frac{k!!}{(k+2l+2)!!}\right)^2 r^{k+2l+2} z^{m-2l}. \tag{9}$$

It is also known that the polynomial basis with degree $\leq s$ in 2D can be written as

$$\begin{aligned} P_s &= \left\{ r^{k-m} z^m : 0 \leq m \leq k, 0 \leq k \leq s \right\} \\ &= \left\{ 1, r, z, r^2, rz, z^2, \ldots, r^s, r^{s-1}z, r^{s-2}z^2, \ldots, rz^{s-1}, z^s \right\}. \end{aligned}$$

Note that $L = (s+1)(s+2)/2$ is the number of polynomial basis functions in P_s. In the one-step MPS, we assume that the solution of (3) and (4) can be represented by

$$u(r,z) \approx \tilde{u}(r,z) = \sum_{i=0}^{s}\sum_{j=0}^{i} \alpha_{ij} \Phi^{ij}(r,z), \tag{10}$$

where

$$\mathcal{L}\Phi^{ij}(r,z) = r^{i-j} z^j, \quad 0 \leq j \leq i, 0 \leq i \leq s. \tag{11}$$

Let $\{(r_i, z_i)\}_{i=1}^{n_i}$ be the interior points in D and $\{(r_i, z_i)\}_{i=n_i+1}^{n}$ be the boundary points on ∂D. Applying (10) to (3) and (4), we have

$$\sum_{i=0}^{s}\sum_{j=0}^{i} \alpha_{ij} \left(r_k^{i-j} z_k^j - \lambda^2 \Phi^{ij}(r_k, z_k) \right) = \hat{f}(r_k, z_k), \quad k = 1, 2, \ldots, n_i, \tag{12}$$

$$\sum_{i=0}^{s}\sum_{j=0}^{i} \alpha_{ij} B\Phi^{ij}(r_k, z_k) = \hat{g}(r_k, z_k), \quad k = n_i + 1, \ldots, n. \tag{13}$$

The above system of equations contains L coefficients to be determined. The number of collocation points n should be chosen larger than L. Once $\{\alpha_{ij}\}$ is determined, the approximate solution $\tilde{u}$ at any point inside the domain can be obtained by (10).