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A local weak form meshless method to simulate a variable order time-fractional mobile–immobile transport model

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ABSTRACT

In this study, a simulation of an anomalous mobile–immobile transport process in complex systems is produced numerically. The process is mathematically described as a two-dimensional time-fractional mobile–immobile advection–diffusion equation in Coimbra variable order derivative sense. A local weak form meshless method combined with a time stepping approach is discussed and implemented to simulate the model. For this purpose, firstly, an implicit difference stepping method is used to semi-discretize the model in time direction. For full discretization, a set of regularly distributed nodes is created in the primary spatial domain and the local radial point interpolation method is used to construct the spatial shape functions on the distributed data-sites. Then an efficient meshless procedure based on combination of local Petrov–Galerkin method and collocation technique is formulated to treat in the interior and on the boundary of the primal spatial domain, respectively. Finally, some benchmark problems are presented on regular and irregular domains to verify the validity, efficiency and accuracy of the method.

1. Introduction

In recent decades fractional calculus has been expanded by scientists as a generalization of classical calculus [1]. Because many of the complicated and anomalous phenomena in physics, chemistry, mechanics and other fields of science and engineering could be successfully described by using the theory of fractional calculus, it has attracted the great attention as an important and powerful mathematical tool to model the anomalous natural phenomena [2–4]. The great advantage of using the theory of fractional calculus for modeling is that the hereditary and inherent memory properties of several materials and complex dynamic processes can be appropriately described by them. Nevertheless, some recent experimental results and new finding indicate that many practical anomalous processes exhibit inherent properties that change in time or space directions. Fortunately, according to the theory of fractional calculus, the fractional models can be generalized as variable order fractional problems [5].

In the current work a two-dimensional variable order time fractional mobile–immobile advection–diffusion model:

$$\beta_1 \frac{\partial}{\partial t} u(\mathbf{x},t) + \beta_2 D_t^{\alpha(\mathbf{x},t)} u(\mathbf{x},t) = -\boldsymbol{\upsilon}.\nabla u(\mathbf{x},t) + D\Delta u(\mathbf{x},t) + \psi(\mathbf{x},t),$$
$$(\mathbf{x},t) \in \Omega \times (0,\infty), \tag{1.1}$$

with the following initial and Dirichlet boundary conditions

$$\begin{aligned} u(\mathbf{x},0) &= \varphi(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega} = \Omega \cup \partial\Omega, \\ u(\mathbf{x},t) &= \rho(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \ t \geq 0, \end{aligned} \tag{1.2}$$

is investigated numerically. Where, $u(\mathbf{x}, t)$ is unknown function should be determined, $\Omega \subset \mathbb{R}^2$ is spatial domain with the smooth boundary $\partial\Omega$, $\nabla(.) = (\partial(.)/\partial x, \partial(.)/\partial y)$ denotes the gradient differential operator and $\Delta(.) = \nabla(.).\nabla(.)$ is the Laplacian operator. Also β_1 and β_2 are two non-negative constants, $\boldsymbol{\upsilon} = (\upsilon_x, \upsilon_y)$ denotes the advection coefficient vector and D is the diffusion coefficient. Moreover $\psi(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ and $\varphi(\mathbf{x})$ are given sufficiently smooth functions. Also $D_t^{\alpha(\mathbf{x},t)}$ denotes the Coimbra variable-order derivative operator of order $\alpha(\mathbf{x}, t)$ $(0 < \alpha(\mathbf{x}, t) < 1)$ with respect to t which is defined as [6]:

$$D_t^{\alpha(\mathbf{x},t)} u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha(\mathbf{x},t))} \int_{0^+}^{t} (t-\eta)^{-\alpha(\mathbf{x},t)} \frac{\partial u(\mathbf{x},\eta)}{\partial \eta} d\eta + \frac{(u(\mathbf{x},t_{0+}) - u(\mathbf{x},t_{0-}))t^{-\alpha(\mathbf{x},t)}}{\Gamma(1-\alpha(\mathbf{x},t))},$$

where $\Gamma(.)$ is the gamma function. The mobile–immobile advection–diffusion model is a common mathematical tool for describing the random motion of particles suspended in a liquid or a gas that known as Browning motion. Therefore, the advection–diffusion equation is widely

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used to describe and model many phenomena in chemical, physical engineering and earth sciences such as solute transport in rivers, groundwater and ocean currents [7–9]. In recent years, some of the computational techniques have been developed and employed to simulate the anomalous mobile–immobile transport process. Liu and et al. numerically investigated the mobile–immobile advection–dispersion model with Caputo constant order time fractional derivative [10–12]. Zhang et al. [13] introduced a mobile–immobile advection–dispersion model with the Coimbra variable time fractional derivative for describing an anomalous transport model in one-dimensional. They proposed and employed an implicit discretization scheme to investigate the model numerically. Ma and Yang in [14] developed a Jacobi spectral collocation method for solving a variable-order time fractional mobile–immobile advection–dispersion solute transport model. An implicit Crank–Nicolson finite difference scheme has been formulated and used to solve model [15]. Jiang and Liu in [16] developed and implemented a computational technique based on reproducing kernel collocation method for simulating the variable-order time fractional mobile–immobile advection–dispersion equation. Very recently, Tayebi et al. [17], employed a meshless numerical method based on the combination of finite difference scheme and moving least squares approximation for solving two-dimensional variable-order time fractional advection–diffusion equation.

Meshless methods based on the radial basis functions (RBFs) [18] are one of the most efficient and powerful class of computational techniques to numerically solve practical engineering problems [19–21]. In recent years, more and more attention has been paid to the meshless numerical techniques based on the RBFs due to their high flexibility and good performance to deal with practical high dimensional models with complicated and irregular domains. Various types of meshless methods based on the RBFs have been introduced and developed, although they are commonly classified into two main categories: meshfree techniques based on the strong form of governing mathematical models [22–28] and meshfree techniques based on the weak form of governing mathematical models [29–35]. Recently, meshless methods based on the radial basis functions in both classes have been developed and widely used for numerically solving various types of constant and variable order fractional differential equations. Chen et al in [36] employed a meshless method based on the unsymmetric RBF collocation method to solve a fractional diffusion problem in two dimensional. A numerical method based on the combination of implicit finite difference scheme and RBF collocation method has been formulated and used for solving a time fractional diffusion equations [37]. Hosseini et al. implemented a RBF collocation method coupled with a time stepping scheme for numerically solving a classical type of time-fractional telegraph equation [38]. In [39], a numerical technique based on the radial basis functions is formulated an employed to simulate a fractal mobile–immobile transport model. In [40], a radial basis function meshless collocation method has been implemented to deal with a space-fractional advection–dispersion equations, which describes an anomalous solute transport process. Also, in [41] the method of approximate particular solutions using the radial basis functions has been used to investigate the fractional diffusion problem with constant and variable order time fractional derivatives. Recently, some of the RBFs meshless techniques in both strong and weak forms have been developed and used by Dehghan et.al, to solve various kinds of fractional differential equations [42–45]. Roohani Ghehsareh et al. in [46] and [47] formulated and implemented a RBF collocation method and a RBF meshless technique based the local Petrov–Galerkin approach to solve the two-dimensional fractional evolution and time fractional cable equations, respectively. Recently, Shivanian et al. developed a spectral meshless radial point interpolation (SMRPI) method for solving fractional problems [48–51].

In the current work, an implicit time discretization scheme combined with an efficient RBf meshless method based on the local weak form of the governing mathematical problems (1.1) and (1.2) would be formulated and implemented for solving the model.

2. Time discretization procedure

Here, an accurate implicit time stepping procedure is formulated to discretize the governing fractional model (1.1) in time direction. For this purpose, firstly, the time interval $[0, T]$ is partitioned into M equal sub-intervals $\bigcup_{n=0}^{M-1}[t^n, t^{n+1}]$, uniformly, where $t^n = n\tau$, $n = 0, \dots, M$ and $\tau = T/M$, denotes the time step size. So clearly the governing Eq. (1.1) is hold at any time level t^{n+1}, as follows:

$$\beta_1 \frac{\partial}{\partial t}u(\mathbf{x}, t^{n+1}) + \beta_2 D_t^{\alpha(\mathbf{x},t^{n+1})}u(\mathbf{x}, t^{n+1}) \\ = -(v_x, v_y).\nabla u(\mathbf{x}, t^{n+1}) + D\Delta u(\mathbf{x}, t^{n+1}) + \psi(\mathbf{x}, t^{n+1}). \tag{2.3}$$

The time integer derivative $\frac{\partial}{\partial t}u(\mathbf{x}, t^{(n+1)})$ can be discretized at three sequential time levels $n+1$, n and $n-1$ as follows:

$$\frac{\partial u}{\partial t}(\mathbf{x}, t^{(n+1)}) = \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} + R_1^{n+1}, \tag{2.4}$$

where $u^n = u(\mathbf{x}, t^n)$. Also, R_1^{n+1} denotes the truncation error that is bounded by

$$|R_1^{n+1}| \le C\tau^2 \tag{2.5}$$

Moreover, for $u(.,.) \in C^2(\Omega \times (0, \infty))$, the Coimbra variable-order derivative $D_t^{\alpha(\mathbf{x},t^{n+1})}u(\mathbf{x}, t^{n+1})$ can be approximated by the following partitioning:

$$\begin{aligned} D_t^{\alpha(\mathbf{x},t^{n+1})}&u(\mathbf{x}, t^{n+1}) \\ &= \frac{1}{\Gamma(1-\alpha(\mathbf{x}, t^{n+1}))}\int_{0^+}^{t^{n+1}} (t^{n+1}-\eta)^{-\alpha(\mathbf{x},t^{n+1})}\frac{\partial u(\mathbf{x},\eta)}{\partial \eta}d\eta \\ &= \frac{1}{\Gamma(1-\alpha(\mathbf{x}, t^{n+1}))}\sum_{j=0}^{n}\int_{j\tau}^{(j+1)\tau} (t^{n+1}-\eta)^{-\alpha(\mathbf{x},t^{n+1})}\frac{\partial u(\mathbf{x},\eta)}{\partial \eta}d\eta \\ &= \frac{1}{\Gamma(1-\alpha(\mathbf{x}, t^{n+1}))}\sum_{j=0}^{n}\left(\frac{u^{j+1}-u^j}{\tau} + O(\tau)\right)\int_{j\tau}^{(j+1)\tau} (t^{n+1}-\eta)^{-\alpha(\mathbf{x},t^{n+1})}d\eta, \end{aligned}$$

moreover, we have,

$$\int_{j\tau}^{(j+1)\tau} (t^{n+1}-\eta)^{-\alpha(\mathbf{x},t^{n+1})}d\eta \\ = \frac{\tau^{1-\alpha(\mathbf{x},t^{n+1})}}{1-\alpha(\mathbf{x}, t^{n+1})}\left[(n-j+1)^{1-\alpha(\mathbf{x},t^{n+1})} - (n-j)^{1-\alpha(\mathbf{x},t^{n+1})}\right].$$

So the following relation is obtained after rearranging:

$$D_t^{\alpha(\mathbf{x},t^{n+1})}u(\mathbf{x}, t^{n+1}) = \frac{\tau^{-\alpha(\mathbf{x},t^{n+1})}}{\Gamma(2-\alpha(\mathbf{x}, t^{n+1}))}\sum_{j=0}^{n}(u^{n-j+1} - u^{n-j})\omega_j^{n+1} + O(\tau^{2-\alpha(\mathbf{x},t^{n+1})}), \tag{2.6}$$

where $\omega_j^{n+1} = (j+1)^{(1-\alpha(\mathbf{x},t^{n+1}))} - j^{(1-\alpha(\mathbf{x},t^{n+1}))}$, $(j = 0, 1, 2, \dots, M)$. By substituting Eqs. (2.4) and (2.6) in (2.3) the following relation at $(n+1)$-th time level, for $n = 1, 2, \dots, M-1$ is obtained:

$$\beta_1 \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} \\ + \beta_2 \frac{\tau^{-\alpha(\mathbf{x},t^{n+1})}}{\Gamma(2-\alpha(\mathbf{x}, t^{n+1}))}\left[u^{n+1} - \sum_{j=0}^{n-1}(\omega_j^{n+1} - \omega_{j+1}^{n+1})u^{n-j} - \omega_n^{n+1}u^0\right] \\ = -(v_x, v_y).\nabla u^{n+1} + D\Delta u^{n+1} + \psi^{n+1}.$$

By rearranging the above relation, the following equation is obtained:

$$\left[\frac{3}{2}\beta_1 + \beta_2\mu(\mathbf{x}, t^{n+1})\right]u^{n+1} - \tau D\Delta u^{n+1} + \tau(v_x, v_y).\nabla u^{n+1} \\ = \beta_2\mu(\mathbf{x}, t^{n+1})\sum_{j=0}^{n-1}(\omega_j^{n+1} - \omega_{j+1}^{n+1})u^{n-j} - \beta_2\mu(\mathbf{x}, t^{n+1})\omega_n^{n+1}u^0 \\ + 2\beta_1 u^n - \frac{1}{2}\beta_1 u^{n-1} + \tau\psi^{n+1}, \quad n = 1, 2, \dots, M-1, \tag{2.7}$$

where $\mu(\mathbf{x}, t^{n+1}) = \frac{\tau^{1-\alpha(\mathbf{x},t^{n+1})}}{\Gamma(2-\alpha(\mathbf{x},t^{n+1}))}$.