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Conformal distance-sigmoidal transformation for evaluating 3D nearly singular integrals over triangular elements

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ABSTRACT

Nearly singular boundary integrals are involved in many applications of BEM, such as thin structures and contact problems. Therefore, the accurate evaluation of such integrals is an important aspect during the successful implementation of BEM. Recently, a conformal mapping for triangular elements has been constructed to eliminate the distorted shape influence and applied to deal with singular integrals. In this paper, the conformal mapping is extended to deal with the nearly singular integrals over triangular elements by combination of the distance transformation proposed by Ma and Kamiya. An improved sigmoidal transformation is employed to rearrange Gaussian points in angular direction more judiciously, and the conformal mapping is introduced to eliminate the distorted shape influence for elements with large aspect ratio or peak/obtuse angles. Extensive numerical tests and comparisons for both planar and curved triangular elements are given to demonstrate the high efficiency and competitiveness of the method presented in this paper.

1. Introduction

The implementation of boundary-type methods involves many numerical integrals over lines or surfaces [1–4]. These integrals can be generalized into three categories depending on the relative position from source point to integral elements, i.e. the regular, singular and nearly singular integrals. For the regular integrals, the Gaussian quadrature can be used directly. For the singular cases, several methods have been devised to improve the accuracy of numerical evaluation, such as the regularization technique [5], the singularity subtraction method [6,7] and other methods [8–10]. However, for the nearly singular integrals, a unified and efficient strategy does not seem to be mentioned in current literature despite the extensive efforts. Accurate computation of nearly singular integrals plays an important role in many engineering applications, especially for thin structures [11], unknowns around crack tips [12], the contact and sensitivity problems [13,14].

In this paper, the nearly singular integrals are concerned. Theoretically, nearly singular integrals are regular in nature since the values of the integrand are always limited. The near singularity is caused by the drastically spiked variation of the integrand, thus the standard Gaussian quadrature cannot be used in a straightforward way. In the past decades, various numerical techniques have been proposed to remove or damp out the near singularities, such as the element subdivision techniques [15,16], the regularization methods [17,18], analytical and semi-analytical algorithms [19,20], and various nonlinear transformations [21–32]. The element subdivision techniques are stable and accurate but not recommended because of its inefficient, especially for the case when the source point lies very close to integral element. The regularization methods translate the singular integrals into non-singular ones and can fully remove the singularity, however, the displacement derivatives at the boundary image point should be obtained in advance. The analytical and semi-analytical algorithms are effective but only limited to linear or planar elements. Curved elements must be divided into a large number of linear or planar elements, thus losing efficiency and accuracy.

At present, the most widely used methods are various non-linear transformations, such as the cubic polynomial transformation [21], the PART method [22], the distance transformation [23–25], the sinh transformation [26–29] and the exponential transformation [30–32] etc. The main idea of the aforementioned non-linear transformations is to remove or damp out the near singularity by non-linear transformations before conventional Gaussian quadrature is applied. Non-linear transformations have been employed to deal with nearly singular integrals with different orders in 2D and 3D BEM, and attractive results can be observed.

However, two difficulties would be encountered when evaluating nearly singular integrals for 3D surface elements. First, the precision would decline dramatically when the source point lies very close to the

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boundary of integral element. Schwab et al. [33] noted this point during the evaluation of singular boundary integrals, and Scuderi [34] resplit the sub-triangles which contain angles greater than $2\pi/3$ for planar elements. In this paper, an improved sigmoidal transformation is constructed based on the previous work of Johnston [35], which can rearrange the Gaussian points in angular direction more judiciously. Another problem is the sensitivity to element shape. Taking triangular elements as example, almost all of the examples in current literature are based on unit triangular element. However, for elements with distorted shape, such as elements with large aspect ratio or peak/obtuse angles, the exiting non-linear transformations would be of lower efficiency. In this paper, the conformal transformation derived from Rong's excellent work [36] is introduced to eliminate the distorted shape influence. The two proposed strategies, when combined with existing distance transformation, will generate an efficient and competitive method for nearly singular integrals.

The paper is organized as follows. A general description of nearly singular integrals and the conventional distance transformation are reviewed briefly in Section 2. Section 3 constructs an improved sigmoidal transformation with emphasis on comparisons with the simple one. Section 4 finds out the origin of shape sensitivity and deduces the conformal transformation in details which can eliminate the distorted shape influence. Numerical examples for triangular elements with both regular and irregular shape are presented in Section 5 to demonstrate the high efficiency and competiveness of the method presented in this paper. The paper ends with conclusions in Section 6.

2. Background

2.1. General description of nearly singular integrals

Considering a 3D domain Ω enclosed by boundary Γ, the displacement boundary integral equation for potential problems can be written in terms of the potential u and the flux q as follows:

$$c(\mathbf{y})u(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{1}$$

where $\mathbf{y}$ and $\mathbf{x}$ are the source point and field point, respectively. $c(\mathbf{y})$ is the coefficient depending on the smoothness of the boundary at point $\mathbf{y}$. $u^*(\mathbf{x},\mathbf{y})$ represents the fundamental solution for 3D potential problems, and $q^*(\mathbf{x},\mathbf{y})$ is the derived fundamental solution with respect to unit outward normal $\mathbf{n}$. The detailed forms of $u^*(\mathbf{x},\mathbf{y})$ and $q^*(\mathbf{x},\mathbf{y})$ are given as

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r(\mathbf{x},\mathbf{y})}, \quad q^*(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi r^2(\mathbf{x},\mathbf{y})}\frac{\partial r(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}} \tag{2}$$

where $r(\mathbf{x},\mathbf{y})$ denotes the Euclidean distance between the source point and field point.

After the boundary discretization and transformation to local coordinate (ξ_1,ξ_2), the integrals in Eq. (1) can be generally written as

$$\mathrm{I} = \int_{-1}^{+1}\int_{-1}^{+1}\frac{f(\xi_1,\xi_2)}{r^\chi}\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{3}$$

where χ denotes the order of r in fundamental solutions, $\chi = 1, 2$ and f is a well-behaved function, consisting of the shape function, Jacobian and coefficients from the derivation of the kernels. When the source point is very close to integral element, the denominator r^χ tends to be zero, resulting in nearly singular integrals with different orders, namely, the nearly weak singularity with kernel u^* and the nearly strong singularity with kernel q^*.

2.2. Distance transformation

The first step of distance transformation is to find the projection point $\mathbf{x}^c$ as shown in Fig. 1(a), i.e. the nearest point from the source point to integral element, and the nearest distance is denoted as r_0. Then, the relationship between the distance r and the local polar coordinate (ρ, θ)

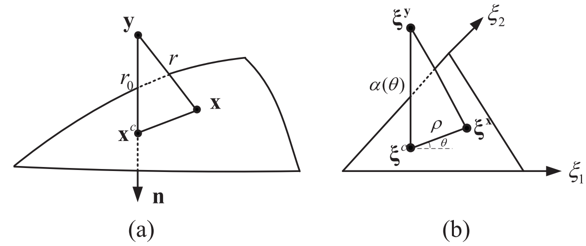

Fig. 1. Distance transformation (a): Global coordinate; (b): Local coordinate.

can be obtained via Taylor expansion in the neighborhood of projection point ξ^c:

$$\begin{aligned} r^2(\rho,\theta) &= r_0^2 + \rho^2 A^2(\theta) + 2r_0 A_k(\theta)n_k(\xi^c) + O(\rho^3) \\ &= A^2(\theta)[\rho^2 + \alpha^2(\theta)] + O(\rho^3) \end{aligned} \tag{4}$$

where

$$\mathbf{A}(\theta) = \frac{\partial \mathbf{x}}{\partial \xi_1}|_{\mathbf{x}=\mathbf{x}^c}\cos\theta + \frac{\partial \mathbf{x}}{\partial \xi_2}|_{\mathbf{x}=\mathbf{x}^c}\sin\theta \tag{5}$$

$$A(\theta) = |\mathbf{A}(\theta)| \tag{6}$$

$$\alpha(\theta) = \frac{r_0}{A(\theta)} \tag{7}$$

Substituting Eq. (4) into Eq. (3) yields

$$\mathrm{I} = \sum_{i=1}^{n}\int_{\theta_1}^{\theta_2}\int_0^{\rho_m(\theta)}\frac{f(\rho,\theta)}{A^\chi(\theta)[\rho^2+\alpha^2(\theta)]^{\chi/2}}\rho\mathrm{d}\rho\mathrm{d}\theta \tag{8}$$

where n is the total number of sub-triangles, and θ_1, θ_2 and $\rho_m(\theta)$ represent the radial and angular span for each sub-triangle, respectively.

For regular triangular elements, the near singularity only occurs in the radial direction and the distance transformation expressed as

$$\eta(\rho,\theta) = \log[\sqrt{\rho^2+\alpha^2(\theta)}] \tag{9}$$

$$\rho(\eta) = \sqrt{\exp(2\eta) - \alpha^2(\theta)} \tag{10}$$

is proposed to deal with the near singularity in Eq. (8). By plugging Eqs. (9) and (10) into Eq. (8), we can obtain

$$\mathrm{I} = \sum_{i=1}^{n}\int_{\theta_1}^{\theta_2}\int_{\ln(\alpha)}^{\ln[\sqrt{\rho_m^2(\theta)+\alpha^2(\theta)}]}\frac{f[\rho(\eta),\theta]}{A^\chi(\theta)[\rho^2(\eta)+\alpha^2(\theta)]^{(\chi-2)/2}}\mathrm{d}\eta\mathrm{d}\theta \tag{11}$$

From Eq. (11), we can see that the nearly singular integrand is transformed into $\frac{1}{A(\theta)}\sqrt{\rho^2+\alpha^2(\theta)}f(\rho,\theta)$ when $\chi = 1$, or $\frac{1}{A^2(\theta)}f(\rho,\theta)$ when $\chi = 2$, which are both well-behaved functions of ρ for elements with regular shape. Therefore, the near singularity resulted from ρ can be damped out by distance transformation.

3. Sigmoidal transformation

Using the above-mentioned distance transformation formula, the nearly singular integrals can be evaluated with high accuracy when the projection point ξ^c lies around the center of integral element. However, numerical tests show that the accuracy of nearly singular integral will decline dramatically when the projection point ξ^c gradually approaches the boundary of integral element, which will be proved in Section 5.1.

According to the geometrical relationship in Fig. 2(a), the radial span $\rho_m(\theta)$ can be written as

$$\rho_m(\theta) = \frac{h}{\cos\bar{\theta}} \tag{12}$$

where h is the perpendicular distance from the projection point ξ^c to the opposite side for each sub-triangle, and $\bar{\theta}$ is the angle from the perpendicular to the field point $\xi^{\mathbf{x}}$. When the projection point ξ^c approaches