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The method of fundamental solutions for solving non-linear Berger equation of thin elastic plate

Bin Lei [a], C.M. Fan [b,c], Ming Li [d,*]

[a] School of Civil Engineering and Architecture, Nanchang University, Jiangxi 330031, China
[b] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung 20224, Taiwan
[c] Computation and Simulation Center, National Taiwan Ocean University, Keelung 20224, Taiwan
[d] Big Data Institute, Taiyuan University of Technology, China

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ABSTRACT

In this paper, we utilized the method of fundamental solutions, which is meshless and integral-free, to analyze the non-linear Berger equation for thin elastic plate. Based on the proposed numerical scheme, the deflection can be expressed as the linear combination of the homogeneous solution and the particular solutions. The particular solution, based on the polyharmonic splines, is derived and then the spatial-dependent loading term of the Berger equation can be approximated by the polyharmonic splines. After the particular solution is obtained, the homogeneous solution, which is governed by the homogeneous partial differential equations, can be solved by the method of fundamental solutions. Several numerical examples are adopted to demonstrate the flexibility and robustness of the proposed meshless scheme, especially the irregular plate with spatial-dependent loading function. Furthermore, we also performed the convergence test for various orders of the polyharmonic splines.

1. Introduction

The von Karman equations and the Berger equation are known as the governing equations for large deflection bending [1,2]. Since the Berger equation is decoupled in comparison with the von Karman equations, it has been applied to static and dynamic deflection problems. If the edge of the plate is rigidly clamped or hinged, the Berger equation is easy to apply and reasonable results can be easily obtained by using this equation.

Due to the non-linearity of the Berger equation, the numerical methods are necessary to approximate the solutions. Since the Berger equation was first proposed in 1955, a number of numerical methods have been proposed, such as the boundary integral equation method [3], the boundary element method [4], the dual reciprocity boundary element method [5], the point-matching method [6], the local boundary integral equation method [7], the charge simulation method [8], to name just a few. During the past two decades, the meshless methods have made significant advances for solving various types of science and engineering problems due to the simplicity and effectiveness of many newly-developed numerical algorithms. Among them, in recent years, the radial basis functions (RBFs) have become very popular in the research area of meshless methods. The RBF collocation method was pioneered by Kansa [9] in 1990 and is widely circulated in the science and engineering community. Another major development of RBFs for numerically solving partial differential equations (PDEs) is the use of the method of particular solutions (MPS) and the method of fundamental solutions (MFS). Coupling the MFS with the MPS, the RBF meshless method has stronger mathematical foundation. We notice that most of the methods mentioned at the beginning of this paragraph are related to the boundary integral equations or the boundary element methods. One of the major reasons of using boundary-type methods for solving the Berger equation is that its fundamental solution is available. However, the boundary discretization and the difficulty of singular or hyper-singular integrations involved in these boundary-type methods are very challenging. Furthermore, for inhomogeneous or nonlinear problems, the domain integral is required which makes these traditional boundary-type methods very tedious and inefficient. The MFS-MPS meshless method has been proved to be very effective in dealing with the above issues. In general, when the fundamental solution and the particular solution of a given PDE are available, the problems can be effectively solved. As we know, the fundamental solution of the Berger equation is already known. However, the evaluation of the particular solution is a delicate issue.

The MFS is an effective boundary-type meshless method for solving homogeneous equations. The MFS was first proposed by Kupradze and Aleksidze [10] in 1964. After its numerical implementation was

* Corresponding author.
E-mail address: liming01@tyut.edu.cn (M. Li).

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proposed by Mathon and Johnston [11] in the 1970's, the MFS started attracting attention in the science and engineering community. In the 1980's, Fairweather and Karageorghis [12] extensively applied the MFS for solving various types of elliptic boundary value problems. At this stage, the MFS is only applicable for solving homogeneous PDEs. In the 1990's, Golberg and Chen [13] made a breakthrough by extending the MFS for solving inhomogeneous problems through the use of RBFs and later further extended it to nonlinear and time-dependent problems [14]. Since then, the MFS has re-emerged and attracted great attention in the science and engineering community. Three review papers [12,15,16] have been devoted to the development of the MFS. For inhomogeneous equation, the MPS is required to obtain the approximate particular solution. Once the MPS is adopted as the numerical scheme, the closed-form RBF for particular solution should be derived, which is not trivial. In this paper, we devoted our effort to deriving the closed-form particular solution for the Berger equation using RBFs and then applying the MFS to find the homogeneous solution. We first solved the linear Berger equation, and then extended the proposed method to non-linear case.

This paper is organized as follows. In Section 2, the Berger equation for deflection of thin elastic plate is described. In Section 3, we briefly review the MPS and then derive a closed-form particular solution for the Berger equation through the well-established RBF particular solutions for the Laplacian and the modified Helmholtz operators. Without going through the tedious derivation using the whole Berger differential operator, we can derive the closed-form particular solution by simple algebraic manipulation; i.e., the beauty of our derivation is simple. Even though the fundamental solution of the Berger equation is known, we can obtain the fundamental solution in a similar way as the particular solution. In Section 4, three examples for linear and nonlinear cases are provided to demonstrate the effectiveness of the proposed method. Some concluding remarks are placed in the last section.

2. Governing equation

Let Ω be a simply-connected domain bounded by $\partial\Omega$. The non-linear Berger equation for the deflection, $u(\mathbf{x})$, of the thin elastic plate is expressed as follows:

$$\left(\Delta^2 - \beta^2\Delta\right)u(\mathbf{x}) = \frac{q(\mathbf{x})}{D}, \quad \mathbf{x} \in \Omega, \tag{1}$$

where D is the bending rigidity of plate, and β^2 is the Berger constant. Δ is the Laplacian and $q(\mathbf{x})$ is the transverse pressure.

If the Berger constant is a known constant, the above equation is a linear PDE. The Berger constant is normally related to the derivatives of the deflection and is expressed as follows:

$$\beta^2 = \frac{6}{sh^2}\int\int_s \left[\left(\frac{\partial u(\mathbf{x})}{\partial x}\right)^2 + \left(\frac{\partial u(\mathbf{x})}{\partial y}\right)^2\right] ds, \tag{2}$$

where s is the area of domain and h is the thickness of the plate.

Since the Berger constant must be calculated from the deflection of the plate, $u(\mathbf{x})$, Eq. (1) becomes a non-linear PDE. The governing equation in (1) subjects to the following boundary conditions:

$$\mathcal{B}_1 u(\mathbf{x}) = f_1(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3}$$

$$\mathcal{B}_2 u(\mathbf{x}) = f_2(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the boundary operators, and $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are known functions.

3. Numerical methods

3.1. The method of particular solutions

Although the MFS is a meshless, integral-free, non-singular, and boundary-type method, it can only be used to solve homogeneous problems. There are a number of methods to extend the MFS for solving inhomogeneous problems. The MPS is one of the effective methods for this purpose. First, the solution, $u(\mathbf{x})$, of (1) – (4) is assumed as the linear combination of the particular solution, $u_p(\mathbf{x})$, and the homogeneous solution, $u_h(\mathbf{x})$; i.e.,

$$u(\mathbf{x}) = u_p(\mathbf{x}) + u_h(\mathbf{x}). \tag{5}$$

The particular solution satisfies the inhomogeneous governing equation, but does not necessarily satisfy the boundary conditions; i.e.,

$$\left(\Delta^2 - \beta^2\Delta\right)u_p(\mathbf{x}) = \frac{q(\mathbf{x})}{D}. \tag{6}$$

To obtain the particular solution, the inhomogeneous term in (6) can be approximated by the RBFs. Let $\{\mathbf{x}_i\}_{i=1}^N$ be arbitrary collocation points containing Ω. By the RBF interpolation, we have

$$\frac{q(\mathbf{x}_i)}{D} = \sum_{j=1}^N a_j \phi\left(r_{ij}\right), \quad 1 \le i \le N, \tag{7}$$

where $\{a_j\}_{j=1}^N$ are the weighting coefficients to be determined, $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between the i^{th} node $\mathbf{x}_i$ and the j^{th} node $\mathbf{x}_j$, and $\|\cdot\|$ is the Euclidean norm. This is a system of N equations with $\{a_j\}_{j=1}^N$ unknowns. If RBF ϕ in (7) is positive definite, the solvability of the above system of equations is assured. But there are many useful RBFs which fail to be positive definite. For conditionally positive definite RBFs, one needs to add the polynomial terms to ensure the solvability of the matrix system of (7). Let P_{m-1}^d denote the space spanned by all d-variate polynomials of degree less than or equal to $m-1$, and $\{p_k\}_{k=1}^t$ is a basis where

$$t = \binom{m-1+d}{d}.$$

Then, the basis functions in (7) are augmented to

$$\frac{q(\mathbf{x}_i)}{D} = \sum_{j=1}^N a_j \phi\left(r_{ij}\right) + \sum_{k=1}^t c_k p_k\left(\mathbf{x}_i\right), \quad 1 \le i \le N, \tag{8}$$

$$\sum_{j=1}^N a_j p_k\left(\mathbf{x}_j\right) = 0, \quad 1 \le k \le t, \tag{9}$$

where $\{c_k\}_{k=1}^t$ are the unknown coefficients to be determined. The solvability of the above system of equations is assured. In contrast to the mesh-dependent numerical methods, the interpolation points can be randomly distributed in the domain. Once the coefficients $\{a_j\}_{j=1}^N$ and $\{c_k\}_{k=1}^t$ are obtained, the particular solution and its derivative terms with respect to x and y can be expressed as follows:

$$u_p(\mathbf{x}) = \sum_{j=1}^N a_j \Phi\left(r_j\right) + \sum_{k=1}^t c_k P_k(\mathbf{x}), \tag{10}$$

$$\frac{\partial u_p(\mathbf{x})}{\partial x} = \sum_{j=1}^N a_j \frac{\partial \Phi\left(r_j\right)}{\partial x} + \sum_{k=1}^t c_k \frac{\partial P_k(\mathbf{x})}{\partial x}, \tag{11}$$

$$\frac{\partial u_p(\mathbf{x})}{\partial y} = \sum_{j=1}^N a_j \frac{\partial \Phi\left(r_j\right)}{\partial y} + \sum_{k=1}^t c_k \frac{\partial P_k(\mathbf{x})}{\partial y}, \tag{12}$$

where

$$\left(\Delta^2 - \beta^2\Delta\right)\Phi(r) = \phi(r), \tag{13}$$

$$\left(\Delta^2 - \beta^2\Delta\right)P(\mathbf{x}) = p(\mathbf{x}). \tag{14}$$

The analytical derivation of $P(\mathbf{x})$ from (14) is easy, but the analytical derivation of $\Phi(r)$ from (13) is by no means trivial. In this section, we adopted a simple procedure to derive the closed-form $\Phi(r)$ for the differential operator of the Berger equation. If β^2 is assumed to be a known constant, (13) can be reformulated as follows:

$$\Delta\left(\Delta - \beta^2\right)\Phi(r) = \phi(r), \tag{15}$$