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Method of fundamental solutions and high order algorithm to solve nonlinear elastic problems

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ABSTRACT

In this work, we propose an algorithm, which combines the Method of Fundamental Solutions (MFS) and the Asymptotic Numerical Method (ANM), to solve two-dimensional nonlinear elastic problems. Thanks to the development in Taylor series, nonlinear elastic problem is transformed into a succession of linear differential equations with the same tangent operator. Recognizing that the fundamental solution is not always available, the Method of Fundamental Solutions-Radial Basis Functions (MFS-RBF) is combined with the Analog Equation Method (AEM) to solve these resulting linear equations. Regularization methods such as Truncated Singular Value Decomposition (TSVD) and Tikhonov regularization associated with the L-curve or Generalized Cross Validation (GCV) criterion have been used to control the resulting ill-conditioned linear systems. The efficiency of the proposed algorithm (MFS-ANM) is validated by comparing the obtained results with those of the classical algorithm based on the finite element method (FEM-ANM).

1. Introduction

Method of Fundamental Solutions (MFS) is a meshless method that belongs to the collocation methods. It has been proposed by Kupradze and Aleksidze [1] and approved its efficiency in solving homogeneous partial differential equations. It has been extended to inhomogeneous partial differential equations by using Radial Basis Functions (RBF) [2] to determine the particular solution. The main idea of MFS-RBF consists in representing the solution of the problem as a linear combination of the fundamental solutions with respect to source points located outside the domain and particular solutions with respect to collocation points. Then, the initial problem is reduced to the determination of unknown coefficients of the linear combination. Marin and Daniel [3] applied this technique for Cauchy problem in two-dimensional isotropic linear elasticity and controlled the conditioning of the resulting system of linear algebraic equations by the first-order Tikhonov regularization associated with the L-curve criterion. Karageorghis et al. [4] studied the inverse problem of coupled thermo-elasticity in the static regime. Sun and Marin [5] give an Invariant Method of Fundamental Solutions (IMFS) for solving boundary value problems in two-dimensional isotropic linear elasticity to satisfy the invariance property.

Furthermore, some works have investigated MFS to solve nonlinear elasticity problems. Among them, we note the works of Naffa and Al-Gahtani [6,7] which have used RBF method to solve nonlinear differential equations governing large deflection of thin plates. Recently, Jankowska and Kołodziej [8] have proposed an elasto-plastic study in the framework of MFS. Generally, all these works combine MFS with classical iterative methods as Newton–Raphson one or variants [9–12]. Tri et al. [13–15] have associated MFS to ANM for solving non linear Poisson problems and computing bifurcation branches. Asymptotic Numerical Method (ANM) is a technique developed to compute the solution of nonlinear partial differential equations. It consists in transforming the nonlinear problem into a sequence of linear ones by expanding the unknowns in power series [16]. As the convergence radius limits the Taylor series, a continuation procedure is developed to obtain the whole solution [17]. ANM has proved its robustness and efficiency for nonlinear problems, such as: buckling of thin structures [18,19], Navier–Stokes equations [20,21], unilateral contact mechanics [22–25], plasticity problems [26,27] and other nonlinear problems.

The aim of the proposed work is to extend this technique to nonlinear elasticity problems taking into account large displacements. The strong form of the governing equilibrium equations of solids is adopted. In fact, the system of linear equations generated by Taylor series

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discretized by MFS is generally ill-conditioned. Then we associate MFS with regularization methods based on Singular Value Decomposition (SVD), Tikhonov regularization method [28] and Truncated Singular Value Decomposition TSVD [29]. The regularization parameter is selected using the Hansen's L-curve method [30] and Generalized Cross Validation GCV criterion [31].

The layout of this paper is as follows. In Section 2, we present the governing equations in the nonlinear elasticity and we illustrate the Asymptotic Numerical Method in Section 3. After that, we show how to solve the resulting linear problems by using the coupling (MFS-RBF-AEM) in Section 4 and discussing the regularization tools in Section 5. Numerical examples involving large deformation problems are provided to show the efficiency and accuracy of the proposed algorithm in Section 6.

2. Governing equations

The most suitable formulation for nonlinear elasticity problems is the Lagrangian one. This approach consists in writing the partial differential equations in a known domain. All unknown fields are supposed functions of the position X of the particle in the reference configuration. Let us denote by Ω the domain occupied by a solid structure defining the reference state and by $\partial\Omega$ its boundary. The studied structure is subjected to prescribed displacements U^d and traction T^d on the disjointed complementary parts of the boundary $\partial\Omega_u$ (Dirichlet boundaries) and $\partial\Omega_f$ (Neumann boundaries). The equilibrium equations and the boundary conditions are formulated with respect to a reference configuration. The static problem to be solved is expressed, in the absence of body forces, as follows:

$$\begin{cases} \nabla \,.\, \Pi(X) = 0 & X \in \Omega \\ \Pi(X)\cdot n = \lambda T^d \; X \in \partial\Omega_f & \\ U(X) = U^d \;\; X \in \partial\Omega_u & \end{cases} \tag{1}$$

where Π is the first Piola–Kirchhoff stress tensor associated with a point X of the domain in its reference configuration, $U = x - X$ designates the displacement field, x being the coordinates of a point in the deformed configuration, n is the outward unit normal vector to $\partial\Omega$ and λ is a scalar parameter. Moreover, we consider a linear constitutive relation taking into account of geometric nonlinearities. This relation can be written in the following form:

$$S = C : \gamma, \tag{2}$$

where C represents the fourth order elastic tensor, S is the second Piola–Kirchhoff stress tensor linked to the first Piola–Kirchhoff stress tensor Π by the relation $\Pi = F \cdot S$, such that F is the transformation gradient tensor defined by $F = I + \nabla U$, with I is the second order identity tensor. The tensor γ represents the Green–Lagrange strain tensor defined by:

$$\gamma = \frac{1}{2}({}^tF \cdot F - I) \tag{3}$$

Our study will be limited to two-dimensional structures $U \equiv \{U\} = {}^t < U_1, U_2 >$. Taking into account the properties of the tensors Π, S and γ, we note $\Pi \equiv \{\Pi\} = {}^t < \Pi_{11}, \Pi_{22}, \Pi_{12}, \Pi_{21} >$, $S \equiv \{S\} = {}^t < S_{11}, S_{22}, S_{12} >$ and $\gamma \equiv \{\gamma\} = {}^t < \gamma_{11}, \gamma_{22}, \gamma_{12} >$. By introducing the generalized gradient vector $\{\theta\}$ which is written as $\{\theta\} = {}^t < U_{1,1}, U_{1,2}, U_{2,1}, U_{2,2} >$, Eqs. (1)–(3) are rewritten as:

$$\begin{cases} \{\gamma\} = ([II] + \frac{1}{2}[A(\theta)])\{\theta\} & in \quad \Omega \\ \{S\} = [C]\{\gamma\} & in \quad \Omega \\ \{\Pi\} = ([III] + [B(\theta)])\{S\} & in \quad \Omega \\ [div]\{\Pi\} = \{0\} & in \quad \Omega \\ [N]\{\Pi\} = \lambda\{T^d\} & on \quad \partial\Omega_f \\ \{U\} = \{U^d\} & on \quad \partial\Omega_u \end{cases} \tag{4}$$

Eq. (4) constitutes the strong formulation of the boundary value problem, with the matrices $[A(\theta)]$, $[B(\theta)]$, $[III]$ and $[II]$ are given by:

$$[A(\theta)] = \begin{bmatrix} U_{1,1} & 0 & U_{2,1} & 0 \\ 0 & U_{1,2} & 0 & U_{2,2} \\ U_{1,2} & U_{1,1} & U_{2,2} & U_{2,1} \end{bmatrix}; [B(\theta)] = \begin{bmatrix} U_{1,1} & 0 & U_{1,2} \\ 0 & U_{2,2} & U_{2,1} \\ 0 & U_{1,2} & U_{1,1} \\ U_{2,1} & 0 & U_{2,2} \end{bmatrix};$$

$$[II] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; [III] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

where $U_{i,j} = \frac{\partial U_i}{\partial x_j}$ $(i, j = 1, 2)$ indicates the derivative of the component U_i with respect to jth variable. The behavior matrix $[C]$ for a homogeneous and isotropic elastic material is written as follows:

$$[C] = \frac{\overline{E}}{1 - \overline{\nu}^2}\begin{bmatrix} 1 & \overline{\nu} & 0 \\ \overline{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\overline{\nu}}{2} \end{bmatrix}, \tag{6}$$

where $\overline{E} = E$ and $\overline{\nu} = \nu$ for the plane stress state, $\overline{E} = E/(1 - \nu^2)$ and $\overline{\nu} = \nu/(1 - \nu^2)$ for the plane strain state, E and ν are respectively the Young's modulus and the Poisson's ratio.

3. Asymptotic Numerical Method (ANM)

In this section, we apply the ANM algorithm to solve the nonlinear problem (4). The basic idea of ANM consists in searching the solution branches of the nonlinear problem in the form of a truncated Taylor expansion from a known and regular solution $(\mathbb{U}_0, \lambda_0)$. More details on this procedure are given in the following references [13,16,32]. To facilitate the illustration of the ANM algorithm, we collect all the unknowns into a single vector $\mathbb{U} = {}^t < \Pi, \theta, S, \gamma, U >$. To solve the nonlinear static problem (4), we seek the solution in the form of a truncated Taylor series expansion with respect to a parameter "a" as follows:

$$\begin{Bmatrix} \mathbb{U} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbb{U}_0 \\ \lambda_0 \end{Bmatrix} + \sum_{i=1}^{p} a^i \begin{Bmatrix} \mathbb{U}_i \\ \lambda_i \end{Bmatrix}, \tag{7}$$

where $(\mathbb{U}_0, \lambda_0)$ is a known starting solution and p is the truncation order. By introducing Taylor series (7) in the Eq. (4) and equating like powers of "a", we obtain the following set of linear problems at each order:

For order: $i = 1$

$$\times \begin{cases} \{\gamma_1\} = ([II] + [A(\theta_0)])\{\theta_1\} \\ \{S_1\} = [C]\{\gamma_1\} \\ \{\Pi_1\} = ([III] + [B(\theta_0)])\{S_1\} + [\widehat{S}_0]\{\theta_1\} \\ [div]\Big(\big([F][C][H] + [\widehat{S}_0]\big)\{\theta_1\}\Big) = 0 \\ [N]\Big(\big([F][C][H] + [\widehat{S}_0]\big)\{\theta_1\}\Big) = \lambda_1\{T^d\} \\ \{U_1\} = 0 \end{cases} \tag{8}$$