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Fast multipole method for poroelastodynamics

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ABSTRACT

Wave propagation phenomena occur in reality often in semi-infinite two-phase (porous) regions. It is well known that such problems can be handled well with the poroelastodynamic Boundary Element Method (BEM). But, it is also well known that the BEM with its dense matrices becomes prohibitive with respect to storage and computing time. This is especially true for poroelastodynamics, where in the best case four degrees of freedom per node are required. As well, the fundamental solution of poroelastodynamics is computationally expensive.

Here, a fast multipole BEM is proposed to circumvent those points. The Chebyshev interpolation-based FMM significantly reduces the memory consumption of the system matrix and thus allows for larger problem sizes to be treated. As well, it requires fewer evaluations of the fundamental solution. To employ an iterative solver, the use of a transformation of the material data is mandatory. Numerical tests show the expected almost linear complexity of the proposed method.

1. Introduction

Porous media occur frequently in nature as well as construction materials. Probably, the best-known example for a natural porous medium is soil. The understanding of wave propagation in such media is of distinct interest for oil and gas explorization but also for earthquake analysis. In these particular fields the treatment of unbounded domains, such as a halfspace, is required. The Boundary Element Method (BEM) is advantageous over the Finite Element Method (FEM) for the numerical treatment of such geometries, since it requires only the discretization of the boundary and inherently fulfills the radiation condition. A review of poroelastic models and their numerical treatment is given in [40].

Based on the work of von Terzaghi, a theoretical description of porous materials saturated by a viscous fluid was presented by Biot [5]. In the following years, Biot extended his theory to anisotropic cases [6] and also to poroviscoelasticity [9]. To treat wave propagation a dynamic theory is necessary, which can be found in two papers, one for low frequency range [7] and the other for high frequency range [8]. Alternatively, the theory of porous media (TPM) can be used, which goes back to Fillunger and has been brought to a mature state by de Boer [15] and Ehlers [18,19]. The dynamic extension of the TPM can be found in [16]. Assuming a linear geometrical and a linear constitutive model, a comparison of both theories has been published by Schanz and Diebels [41], which shows that the mathematical operator of both theories is the same. This has the physical consequence that in both theories three waves exist, two compressional waves and one shear wave. The numerical consequence is that all methods can be transferred from one theory to the other as long as the linear description holds.

As already written above, here the focus is on a boundary element formulation. First poroelastodynamic BE formulations based on Biot's theory have been published in Laplace domain by Manolis and Beskos [30,31] expressed in terms of solid and fluid displacements. However, it can be shown that only the solid displacements and one additional variable, the fluid pressure, are independent [10]. Formulations in frequency domain have been published by Cheng et al. [14] and Domínguez [17] based on these unknowns. A time domain formulation was developed by Wiebe and Antes [48], but with the restriction of vanishing damping between the solid skeleton and the fluid. Another time dependent formulation was proposed by Chen and Dargush based on analytical inverse transformation of the Laplace domain fundamental solutions [13]. Utilizing the convolution quadrature method [28,29] a time stepping based BE formulation has been proposed by Schanz [38,39]. The latter formulation uses the Laplace domain fundamental solutions but works in time domain. Hence, all regularization techniques known for the elliptic operator can be applied, which has been done in [32]. All the above mentioned formulations are based on the first integral equation and the collocation method. A symmetric Galerkin based formulation has been presented by Messner and Schanz [33] and a mathematical analysis of the Laplace domain version can be found in [43].

The above mentioned BE-formulations have a quadratic complexity in the spatial variable. Hence, for real world problems the effort becomes prohibitive. To overcome this drawback so-called fast methods

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have become popular in the field of applied mathematics and engineering. The history of such methods, i.e., asymptotically optimal approximations of dense matrices, starts with the paper by Rokhlin [37]. For the first time an algorithm was presented which scales like $\mathcal{O}(n \log n)$. Subsequently, the so called Fast Multipole Method (FMM) has been developed in [22] for some large-scale n-body problems. The method was significantly improved in [23]. In the work of Of et al. [36] the FMM is applied to elastostatic problems based on a Galerkin BEM discretization. The extension to elastodynamics in Fourier domain has been published in [12] based on a collocation approach. In time domain, the FMM with a plane wave expansion is presented in [44]. A textbook on the FMM with application in collocation BEM has been published by Liu [27] and a literature review can be found in [35].

Another approach is the wavelet based BEM [1], which produces sparse matrices based on orthogonal systems of wavelet like functions. On a purely algebraic level works the Adaptive Cross Approximation (ACA), which has been proposed by Bebendorf [2], Bebendorf and Rjasanow [4]. The latter fast method is often classified as black-box method, because the ACA is (nearly) independent of the kernel function. An application to the Galerkin BEM in elastostatics can be found in [3]. Other black-box methods for a FMM uses a series expansion of the kernel and can in this sense also be classified as black-box. The Panel Clustering (see [24]) is the first of these methods. A refined version, where the kernel expansion is based on a Chebyshev interpolation, has been published by Fong and Darve [21]. The extension of this approach to acoustics with a directional clustering is presented in [34]. The advantage of these black-box methods against the usual FMM is that no analytical kernel expansion has to be known. However, the price to pay is usually a higher complexity which is still almost linear but the expansion order of the kernel enters with a power of six in 3-d instead of four using, e.g., spherical harmonics for the Helmholtz kernel. But for poroelastodynamics this black-box property is essential as no analytic kernel expansion seems to be possible. There might be a possibility to reduce the kernel to a combination of derivatives of the Helmholtz kernel as it has been done in [12] for elastodynamics. However, this requires to shift some derivatives to the interpolation polynomials, which increases the necessary polynomial order, and a lot of analytical preparations of the kernel. A study on these different approaches can be found for elastodynamics in the thesis [45].

Here, a FMM based on a Chebyshev interpolation will be presented for poroelastodynamics in Laplace domain. For an uncoupled quasistatic poroelastic problem the same FMM idea has been applied in [47]. The presented Laplace domain formulation includes also problems in Fourier/frequency domain if the real part of the Laplace variable is set to zero. The idea is to use the whole vectorial kernel within the Chebyshev interpolation of the kernel. This results in vectorial M2L-operators but avoids to high interpolation orders. Further, the method is nearly black-box and can use already coded functions for the evaluation of the kernel.

The paper is organized as follows. First, the basic equations and the corresponding integral equations are recalled. The FMM is given in Section 3 in a short way because it is only an extension from the algorithm given by Fong and Darve [21]. The modifications necessary for the vectorial kernel of poroelastodynamics will be discussed. The proposed formulation is then tested in Section 4.

Throughout this paper, vectors and tensors are denoted by bold symbols and matrices and vectors of the discretized system by upper case and lower case sans serif symbols, respectively. No summation convention is used in the entire work. The indices of a matrix $(\mathsf{A})_{ij}$ indicate the ijth entry, which might be a scalar or a matrix value.

2. Governing equations

Following Biot's approach to model the behavior of a porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered [6]. This porosity is denoted by ϕ, which is the relation of the volume of the interconnected pores to the bulk volume. Contrary to these pores the sealed pores will be considered as part of the solid. Full saturation is assumed, i.e., a two-phase material is given. Assuming compressible constituents, a linear elastic solid skeleton, and a linear geometrical description results in a set of coupled partial differential equations (for details see [7]). The problem can be given as follows.

2.1. Problem setting

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\Gamma := \partial\Omega$ its boundary with the outward normal $\mathbf{n}$. The coupled set of homogeneous partial differential equation for the solid displacements $\mathbf{u}(\mathbf{x})$ and the pore pressure $p(\mathbf{x})$ are considered

$$\left.\begin{aligned} G\nabla\mathbf{u}(\mathbf{x}) + \left(K+\frac{1}{3}G\right)\nabla\nabla\cdot\mathbf{u}(\mathbf{x}) - s^2\left(\varrho-\beta\varrho_f\right)\mathbf{u}(\mathbf{x}) - (\alpha-\beta)\nabla p(\mathbf{x}) &= 0 \\ \frac{\beta}{s\varrho_f}\nabla^2 p(\mathbf{x}) - \frac{\phi^2 s}{R}p(\mathbf{x}) - (\alpha-\beta)s\nabla\cdot\mathbf{u}(\mathbf{x}) &= 0 \end{aligned}\right\} \quad \forall\mathbf{x}\in\Omega \tag{1}$$

with the parameter

$$\beta = \frac{\kappa\varrho_f\phi^2 s}{\phi^2 + s\kappa\left(\varrho_a + \phi\varrho_f\right)} \quad \text{and} \quad \varrho_a = 0.66\,\phi\varrho_f .$$

It is assumed that the pore pressure and the solid displacements are in Laplace domain with the Laplace variable $s \in \mathbb{C}$ s.t. $\Re s > 0$. The used material data are the shear and bulk modulus of the skeleton, G and K. The bulk density $\varrho = (1-\phi)\varrho_s + \phi\varrho_f$ is the weighted combination of the solid density ϱ_s and the fluid density ϱ_f. The Biot parameters are denoted by R and α, where the latter describes the compressibility of the solid grains (i.e., $\alpha = 1$ is the incompressible limit). The spatial derivatives are given with the ∇-operator with its usual meanings as $\operatorname{grad} = \nabla$ or $\operatorname{div} = \nabla\cdot$. The assumption for the apparent mass density ϱ_a holds for low frequencies [11]. For different materials the factor 0.66 might be different and a certain frequency dependency can be added [8]. Further, for higher frequencies as well the permeability should be modeled as a function of s [25], which does not change the subsequent derivation. However, the proposed FMM will be restricted to low frequencies and, consequently, here the dependency is skipped. The boundary is split into non-overlapping sets Γ_D and Γ_N such that $\Gamma = \Gamma_D \cup \Gamma_N$ holds. The Dirichlet and Neumann boundary conditions are given by

$$\begin{aligned} &\mathbf{u}(\mathbf{x}) && = \mathbf{f}_D(\mathbf{x}) && \forall\mathbf{x}\in\Gamma_D \\ &p(\mathbf{x}) && = g_D(\mathbf{x}) && \forall\mathbf{x}\in\Gamma_D \\ &\mathbf{t}(\mathbf{x}) = \mathcal{T}^S\mathbf{u}(\mathbf{x}) - \alpha\mathbf{n}p(\mathbf{x}) && = \mathbf{f}_N(\mathbf{x}) && \forall\mathbf{x}\in\Gamma_N \\ &q(\mathbf{x}) = -\frac{\beta}{s\varrho_f}\left(\frac{\partial}{\partial\mathbf{n}}p(\mathbf{x}) + \varrho_f s^2\mathbf{n}^\top\cdot\mathbf{u}(\mathbf{x})\right) && = g_N(\mathbf{x}) && \forall\mathbf{x}\in\Gamma_N, \end{aligned} \tag{2}$$

where $\mathbf{t}(\mathbf{x})$ is the total traction vector and $q(\mathbf{x})$ the flux. In (2), the elastic traction operator

$$\mathcal{T}^S\bullet = \left(K - \frac{2}{3}G\right)\mathbf{n}\nabla\cdot\bullet + 2G\frac{\partial}{\partial\mathbf{n}}\bullet + G\mathbf{n}\times(\nabla\times\bullet) \tag{3}$$

has been used. The given set of partial differential equations is the so-called $u-p$-formulation, which needs only four (in 3D) degrees of freedom (dofs). There exits also formulations for poroelastic continua with more dofs, which are formulated with the solid and fluid displacements or even with the latter both and the pore pressure. However, for linear problems the proposed one is sufficient and has the minimum amount of dofs [10]. Certainly, the boundary condition type, Dirichlet or Neumann, might differ in each direction of the vectorial dofs and between the elastic and fluid dofs. However, for simplifying notation this is not separately denoted. Also it should be remarked that all variables are in Laplace domain as noted above. A frequency domain formulation is automatically included by setting the real part of s to zero.

As already reported in [13] for a reliable numerical method dimensionless variable should be introduced. Based on the suggestion in [13] a