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Numerical solutions of mild slope equation by generalized finite difference method

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ABSTRACT

The mild slope equation (MSE) has been widely used to describe combined wave refraction and diffraction in the field of coastal and offshore engineering owing to its applicability for a wide range of wave frequencies. In this paper, a meshless numerical algorithm, based on the generalized finite difference method (GFDM), is firstly proposed to efficiently and accurately solve the MSE. As a newly-developed domain-type meshless method, the GFDM can truly get rid of time-consuming meshing generation and numerical quadrature. The partial differential terms of the MSE for each point in the computational domain can be discretized into linear combinations of nearby function values with the moving-least-squares method of the GFDM, so the numerical implementation is very convenient and efficient. To evaluate the accuracy and capability of the proposed scheme for MSE, a series of numerical tests were conducted, covering a range of complexity that included propagation and transformation of waves due to a parabolic shoal, a circular island mounted on a paraboloidal shoal and elliptic shoal situated on a slope, as well as breakwater gap. The results were compared with experimental data, analytical solutions and other numerical methods, and reasonable agreements have been achieved.

1. Introduction

Wave propagation in various water depth usually accompany with refraction, diffraction and other wave transformations. These characteristics of wave motion will have significant influence on applications of coastal engineering, hence it is imperative to estimate the offshore wave conditions for coastal structure designing, sediment transport and economical operation. In 1972, assuming irrotational linear harmonic waves and ignoring energy loss due to friction or breaking, Berkhoff [1] derived the original mild slope equation (MSE) by integrating the Laplace equation over the water depth after multiplying a water depth function. Thereafter, the MSE has been widely used in the field of coastal engineering for its reliability in dealing with complex wave problem and its accuracy of describing combined refraction and diffraction phenomenon.

The original MSE is a single frequency wave equation based on linear wave theory, so many additional physical effects, which play essential roles in the wave transformations prediction, are not taken into account. To describe more realistic complex wave transformations, the original MSE has been improved by many researchers with different ways, such as taking into account the terms of fractional dissipation [2–4] and of steep slope [4–6], considering wave breaking [7,8] and wave-current interaction [9,10], as well as extending the original MSE to include time element forming a time-dependent equation [11–13].

Since the equation is essentially the elliptic type with inseparable characteristics, presenting a direct solution to the original MSE is problematic. Therefore, various numerical methods have been proposed to solve the MSE. The iterative method, based on the conjugate-gradient (CG) technique, was developed by Panchang et al. [14], who used a preconditioning method to accelerate the speed of convergence in the solution process of solving the MSE. Moreover, a generalized conjugate method without using the preconditioning method was proposed by Li [15] to solve the MSE and achieved a result as good as Panchang's model. Tang et al. [16] combined the finite difference method (FDM) and the generalized product-type bi-conjugate gradient (GPBiCG) method to simulate wave propagating in the near shore region. Chen et al. [17] used a finite element coastal wave method to simulate the wave-current interaction phenomenon by solving an extended mild slop wave current equation. Besides, a numerical model is developed by Liu et al. [18] based on the preconditioned self-adaptive finite element model (FEM) to solve the MSE. In addition, Naserizadeh et al. [19] used the boundary element method (BEM) and a high-order FDM conjunctively to solve the modified MSE. However, each of these methods has advantages and disadvantages for the applicability, accuracy and stability, as well as computational cost when dealing with different equation.

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Over the past few decades, the conventional mesh-based methods have been developed quite perfectly. Nevertheless, in some cases such as the complex-geometry and higher dimensional problems, mesh generation and numerical integration need considerable amount of time and thus decrease computational efficiency greatly. Being regarded as a promising alternative to classical mesh-based methods, the meshless methods have the potential to avoid the problem of domain or surface grid generation and numerical quadrature. Since it was put forward, the meshless numerical schemes have been gradually derived a series of methods such as the method of fundamental solutions (MFS) [20,21], the smoothed-particle hydrodynamics (SPH) [22], the element-free Galerkin method (EFGM) [23], the modified collocation Trefftz method (MCTM) [24,25], the meshless local Petrov–Galerkin method (MLPGM) [26], the local radial basis function collocation method (LRBFCM) [27] and the generalized finite difference method (GFDM) [28–38].

Within the above meshless methods, the GFDM belongs to a domain-type one, since both of boundary nodes and interior nodes are simultaneously adopted in numerical implementation. In the GFDM, by utilizing the weighted least squares fitting technique and Taylor series expansion, the derivatives of the unknown variables for each point in the computational domain can be discretized into linear combinations of nearby function values with different weighting coefficients, then a sparse system of nonlinear algebraic equations is yielded and can be efficiently solved by using various sparse matrix solvers. This feature makes the GFDM easy-to-program, straightforward and efficient when applied to a large and complicated computational region. Benito et al. [28] derived the explicit formulas of the GFDM and some influencing factors on numerical accuracy were discussed with several sets of mathematical cases. Thereafter, Gavette et al. [31] improved the GFDM and obtained satisfying results by comparing with other meshless methods. For over a decade, the GFDM has been gradually applied on various mathematical problems. Benito et al. [29,30] applied the GFDM to solve parabolic and hyperbolic equations and improved the approximated solution of partial difference equations. More challenging, the GFDM is used to solve third-and fourth-order partial differential equations by Urena et al. [32]. Recent advances, which enable approximation of nonlinear conditions, have the potential to extend the GFDM for various scientific and engineering applications. Chan et al. [33] utilized the GFDM and a newly-developed solver for nonlinear algebraic equation to deal with two-dimensional nonlinear obstacle problems, while Fan et al. [34,35] applied the GFDM on inverse biharmonic boundary-value problems and two-dimensional Cauchy problems. Zhang et al. [36,37] adopted the GFDM to simulate the two dimensional sloshing phenomenon and the propagation of nonlinear water waves in numerical wave flume. Li and Fan [38] utilized the GFDM to analyze the two-dimensional shallow water equation.

In this paper, we investigated a meshless numerical scheme, based on the GFDM, for solving the wave transformation processes in nearshore region, including diffraction, refraction, reflection, and weak nonlinearity, which is governed by the MSE. The proposed GFDM-based model is truly free from mesh generation and numerical quadrature, so it is very effective, simple and accurate to deal with problems in irregular domains, governed by the MSE. It should be noted that, as described above, the GFDM has been recently applied to various partial differential equations (PDEs). To the best of our knowledge, this is the first time that the GFDM is applied for accurately solving the MSE. In addition, this paper focuses on numerical solution of nonlinear equations, although many published GFDM-related papers mainly investigated numerical solutions of linear PDEs. Besides, the flexibility of the GFDM is also emerged since the spatial derivatives at any position can be easily acquired by the GFDM. It is worthwhile to mention that, for much shorter period of incident wave, the computational cost for most numerical method will substantially increase due to sufficient accuracy. This situation is a challenging test for any numerical model. A surprising result is obtained in the present paper. Furthermore, the accuracy of the MSE models is carefully verified in the way for linear and nonlinear dispersion relations. Four numerical examples, including refraction of long waves over a parabolic shoal, wave around a circular island on a paraboloidal shoal and wave propagation over an elliptic shoal, as well as single gate breakwater, are provided to assess the merits of using the GFDM for numerical solutions of the MSE. In addition, some correlation parameters are investigated to verify the stability and convergence of the proposed meshless numerical scheme.

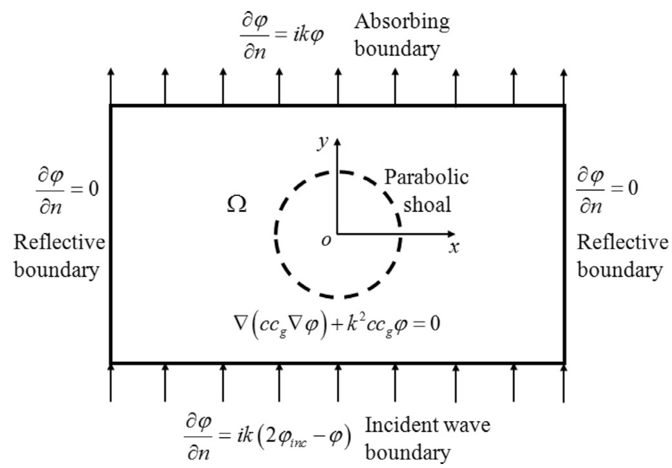

Fig. 1. Schematic diagram of computational domain and boundary conditions in this study.

2. Governing equation and boundary conditions

2.1. Governing equations

When a homogeneous incompressible fluid with irrotational motion travels over a sea bottom with variable depth $h(x, y)$, diffraction and reflection usually occur due to shoals or solid boundaries. Thus, the fluid can be expressed in terms of the velocity potential $\Phi(x, y, z, t)$, which satisfies the Laplace equation as follow [1],

$$\nabla^2\Phi + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad -h(x, y) \le z \le 0 \tag{1}$$

The linear free surface boundary conditions for harmonic waves is,

$$\frac{\partial\Phi}{\partial z} + \frac{\omega^2}{g}\Phi = 0 \qquad z = 0 \tag{2}$$

The kinematic boundary condition at the impermeable bottom is,

$$\frac{\partial\Phi}{\partial z} + \nabla h \cdot \nabla\Phi = 0 \qquad z = -h(x, y) \tag{3}$$

Where x, y denote the horizontal coordinates while z is the vertical coordinate measured positively upwards with the undisturbed free surface at $z=0$. $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is used to indicate the horizontal gradient operator and g denotes the gravitational acceleration.

For monochromatic waves, the velocity potential $\Phi(x, y, z, t)$ can be presented as [39],

$$\Phi(x, y, z, t) = \mathrm{Re}\left\{\varphi(x, y) f(z) e^{-i\omega t}\right\} \tag{4}$$

in which symbol *Re* represents the real part of a complex value and φ is the two-dimensional complex horizontal wave velocity potential of the water surface. $f(z)$ is the depth dependency, provided by

$$f(z) = \frac{\cosh\left[k(h+z)\right]}{\cosh(kh)} \tag{5}$$

where $k = k(x, y)$ denotes the local wave number.

Referring to Fig. 1, a Cartesian coordinate system is adopted, in which the horizontal coordinate (x, y) is located at the still water