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# Degenerate scale for 2D Laplace equation with mixed boundary condition and comparison with other conditions on the boundary

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ABSTRACT

It is well known that the 2D Laplace Dirichlet problem has a degenerate scale for which the direct boundary integral equation has several solutions. We study here the case of the mixed boundary condition, mainly for the exterior problem, and show that this problem has also one degenerate scale. The degenerate scale factor is a growing function of the part of the boundary submitted to Neumann condition. Different special cases are then addressed: segment, circle and symmetric problems. Some exact values of the degenerate scale factor are given for equilateral triangle and square. The numerical procedure for determining the degenerate scale factor for mixed BC is described. The comparison is made with other kinds of boundary conditions and the consequence of the choice of Green's function when using the Boundary Element Method is studied.



## 1. Introduction

Since its early development, the Boundary Element Method (BEM) method has been intensively investigated. Notably, the error of the BEM has been evaluated [1–3] and new numerical improvements are still being suggested using adaptative mesh [4–7], isogeometric boundary element analysis with non-uniform rational B-splines [8–10] and adaptative cross approximation [11] or reducing the singularities of the boundary integral formulation [12,13].

However, the issue of the degenerate scale for 2D problems persists because it is linked to the underlying Boundary Integral Equation (BIE). The existence of a degenerate scale for the Laplace equation with Dirichlet boundary condition (BC) in the plane is well known [14,15]: for a special size of the domain under study, characterized by its scale when compared with all homothetic domains (the "degenerate scale"), the BIE has more than one solution for Dirichlet BC over all the boundary. Numerous results were obtained on the Laplace problem with Dirichlet BC, either from a fundamental point of view, in relation with potential theory [16–23] or for application to BEM [24–34]. Specific investigations for Laplace's problem deal with the way to evaluate numerically the degenerate scale [27,35], to mitigate the numerical issues coming from degenerate scale [24,26,27,33,36] or to deal with special geometrical configurations [34,37–40] or multiply connected problems [29,30,41]. Some of these results were extended to the case of plane elasticity [42–48] and to biharmonic equation [49,50]. For conduction problems, it has been shown that degenerate scale for anisotropic conduction can be obtained from the ones obtained in the case of isotropy [51]. The Laplace problem with Dirichlet BC has been extended to the case where the Green's function corresponds to the half plane [52] and to Robin boundary condition [53]. It is also worthwhile mentioning that the question arises also in dynamics: under special conditions, the degenerate scales arise also in solving the Helmholtz equation [54], because the dynamic singular kernel is asymptotically logarithmic. From a general point of view, degenerate scales appear because of the logarithmic part of the kernels that are involved when solving plane problems using a Boundary Integral Equation, which is the case for conduction, elasticity, elastoplasticity,...

The mixed BC (i.e. when a part of the boundary is at Dirichlet BC while the other part is at Neumann BC) is the main contribution of the present paper. It has been investigated in [19,28], for the interior problem uniquely. It was shown that if the logarithmic capacity [16–18,23,31,55] is equal to 1 (or equivalently if the Dirichlet problem is at its degenerate scale), the interior mixed problem is at a degenerate scale. The existence and uniqueness of the solution of the integral equations for the interior problem are proved in [56] with the condition that the diameter of the domain is less than 1; this paper also contains an extensive bibliography of early works on mixed boundary problems.

The aim of this paper is to complete the results on degenerate scale by providing a thorough study of the case of mixed BC. We focus on the direct method of BIE (see for example [57]). In a first step, we recall the main results in the case of boundary conditions which are of the same kind over all the contour. Next, we complete the known results

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in the case of the interior boundary value problem by showing that the degenerate scale for Dirichlet BC is the unique degenerate scale for the interior problem with mixed BC. Then, we address the exterior problem. One difference compared with the case of the interior problem is that the non unique solution of the BIE leads to a non null solution of the boundary problem via the representation formula. There is a unique degenerate scale for this exterior mixed problem and the degenerate scale factor increases when the part of the boundary submitted to Neumann BC increases. The special case of segments is dealt with. Then some special cases are solved using conformal mappings and symmetries, leading to exact solutions for some cases of equilateral triangle and square. The numerical procedure for obtaining the degenerate scale is provided and finally, we compare the degenerate scale for mixed BC with other cases where the BC is the same over all the boundary: Dirichlet, Robin, Neumann.

## 2. Principal results on degenerate scales for BC of the same type over the contour

This section reminds shortly the most important results on degenerate scales in the case of Dirichlet, Robin and Neumann BC. It aims to prepare the study of mixed BC which is an analogous issue, needing often similar methods, and to allow the final comparison for all boundary conditions in Section 11.

### *2.1. Dirichlet BC*

The case of Dirichlet BC over all the contour has been the object of numerous studies. The degenerate scale is related to the nature of the BIE in the case of plane problems, which contains the integral operator over the boundary $\Gamma$

$$F_q(x) = \int_\Gamma q(y)G(x,y)\mathrm{d}S_y, \tag{1}$$

Where $G$ is the Green function for the plane, $G = \frac{-1}{2\pi}\ln(||x-y||)$. The problem is to find the function $q$ when the function $F_q(x)$ is known over $\Gamma$.

$F_q(x)$ is known from the boundary condition in the case of Dirichlet BC, either in the case of direct or indirect formulation of the BIE, while the meaning of $q$ differs between these two formulations. In the case of the Direct formulation of BIE, it is given by:

$$F_q(x) = \frac{1}{2}u(x) + \int_\Gamma u(y)\frac{\partial G(x,y)}{\partial n(y)}\mathrm{d}S_y, \tag{2}$$

In this formulation, $q$ is the unknown boundary normal flux.

The degenerate scale is obtained by studying all domains corresponding to homothetic contours $\rho\Gamma$ of $\Gamma$. It was shown that this operator can be null for non null values of $q$ and a specific boundary $\rho_0\Gamma$ which is homothetic to $\Gamma$ by a factor $\rho_0$. This specific value of $\rho_0$ is the degenerate scale factor. This factor is related to the logarithmic capacity $C$ [16–18,23] by $\rho_0 = 1/C$. This result is important since the logarithmic capacity of numerous kinds of contours is known, that allows to find easily the degenerate scale, a domain being at the degenerate scale if its logarithmic capacity is null. The search of degenerate scale does not involve the normal to the boundary. As a consequence, interior and exterior problems with Dirichlet BC related to the same boundary correspond to the same degenerate scale.

The existence of degenerate scales has an important consequence when using the BEM, which rests on the discretized version of the boundary integral equation, i.e. the linear system:

$$[H][u] = [G][q] \tag{3}$$

where $[u]$ contains the values of the searched harmonic potential at nodes located at the boundary and $[q]$ the values of the normal gradient of the potential at the same nodes. The numerical degenerate scale corresponds to the size of the domain for which the matrix $[G]$ is singular. The consequence of the existence of a degenerate scale is that the matrix $[G]$ becomes badly conditioned for domains whose scales are near the one of the boundary at the degenerate scale [28].

### *2.2. Convection type or Robin BC*

A convection type BC has the form :

$$\frac{\partial u}{\partial n} = -t\,(u-u_0), \tag{4}$$

where $u_0$ is given over the boundary and $t$ is a positive physical constant related to the convection coefficient and to the conductivity. Even if this condition is well known in the case of thermal condition, it can be used in other physical cases related to conduction: it corresponds to the existence of a very conductive thin layer over the boundary. Assuming that this boundary condition is applied over all the boundary, the boundary integral equation over $\Gamma$ can be written as:

$$\frac{1}{2}u(x) + \int_\Gamma G(x,y)\,t\,u(y)\mathrm{d}S_y + \int_\Gamma \frac{\partial G}{\partial n_y}u(y)\mathrm{d}S_y = \int_\Gamma G(x,y)t\,u_0(y)\mathrm{d}S_y. \tag{5}$$

If the domain bounded by $\Gamma$ is at a degenerate scale for this BC, it means that the operator on the left hand side has a non-null solution for a null value of $u_0$. It means that the problem for $u_0 = 0$ in the convection condition has a non null solution. This corresponds to the Robin condition, t being the Robin constant.

The main results concerning the degenerate scale for this problem can be found in [53]. In this paper, it has been shown that, for the interior problem, the degenerate scale is the same as for the Dirichlet problem. However, for the exterior problem, the degenerate scale is now related to the value of the Robin constant.

### *2.3. Neumann BC*

In the case of Neumann BC over all the contour, it is well known that the solution of the interior problem is obtained up to a constant if the compatibility condition $\int_\Gamma \frac{\partial u}{\partial n}\mathrm{d}S_y = 0$ is satisfied (e.g. [58]). There is always a solution of the exterior problem which is unique up to a constant (e.g. [58]). There is no degenerate scale.

## 3. Study of the degenerate scale of interior problem with mixed boundary condition

As seen previously, a degenerate scale does appear in the case of Dirichlet BC and in the case of convection (Robin) BC. However, in practice, different boundary conditions may be mixed and another very important case is the case when a part of the boundary corresponds to Dirichlet BC and the other to Neumann BC. This mixed BC case will be studied in this section for interior problem and in the following sections 4-10 for exterior problems.

### *3.1. Definition of the degenerate scale for interior problem with mixed BC*

We consider the following interior Laplace problem:

$$\begin{cases} \Delta u = 0 \quad x \in \Gamma^+; \\ u(x) = 0 \quad x \in \Gamma_D; \\ \dfrac{\partial u}{\partial n}(x) = 0 \quad x \in \Gamma_N. \end{cases} \tag{6}$$

Such a function satisfies the following boundary equations (e.g. [57]):

$$\begin{cases} \displaystyle\int_{\Gamma_N} u(y)H(x,y)\mathrm{d}S_y - \int_{\Gamma_D} q(y)G(x,y)\mathrm{d}S_y = 0 & x \in \Gamma_D \\ \displaystyle\frac{1}{2}u(x) + \int_{\Gamma_N} u(y)H(x,y)\mathrm{d}S_y - \int_{\Gamma_D} q(y)G(x,y)\mathrm{d}S_y = 0 & x \in \Gamma_N \end{cases} \tag{7}$$

with $q = \frac{\partial u}{\partial n}$, $G(x,y) = -\frac{1}{2\pi}\ln(|x-y|)$, $H(x,y) = \frac{\partial G(x,y)}{\partial n_y}$.