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A fast boundary integral equation method for point location problem

Qiao Wang [a,b], Wei Zhou [a,b,*], Yonggang Cheng [a,b], Gang Ma [a,b], Xiaolin Chang [a,b], E Chen [c]

[a] State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China
[b] School of Water Resources and Hydropower Engineering, Wuhan University, Wuhan 430072, China
[c] Department of Civil and Environment Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

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ABSTRACT

A numerical method based on the boundary integral equation is proposed for the point location problem. For a bounded domain, the integral value is close to 1.0 if a point is inside the domain, and is close to 0.0 when the point is outside the domain. For convenience of integration, the boundary of the domain can be discretized into boundary integral cells. The idea of isogeometric analysis can be easily coupled with the proposed method, i.e., using the parametric functions in geometric modeling to create the integral cells, which results in a mesh-free procedure for which the geometry can be exactly produced at all stages. Thus, the method can be applied to arbitrary shapes and easily embedded in computer-aided design (CAD) packages. The method is time-consuming if implemented directly; a fast multipole method is coupled with the proposed method to accelerate the integral procedure. Some examples of 2D and 3D cases are tested to show the accuracy and efficiency of the proposed method.

1. Introduction

The point location problem [1] is one of the most fundamental operations in computational geometry [2], and it has many applications in computer graphics (CG), computer-aided design (CAD), and computer-aided engineering (CAE). In general, point location deals with the following problem [3]: given a set of disjoint geometric objects, determine the object containing a query point. In other words, this problem can also be defined as: given a set of points and a bounded domain, determine the domain containing which points. This problem can also be found in some numerical methods, such as in element free Galerkin (EFG) method [4], one may need to determine which integral points are in the computational domain while computing the background integrals. In the adaptive cell-based domain integration method (CDIM) [5,6] for evaluating domain integrals in boundary element method (BEM), one also need to overcome this problem.

Many methods have been proposed for the point location problem, including the ray-crossing algorithms [7], the triangle-based algorithms [8,9], the algorithms based on the sum of angles [10], the cell-based method [11], and the trapezoid/BSP structure-based method [12,13]. One of the best-known algorithms may be the ray-crossing (RC) method, which suffers from singularities. The RC test can be accelerated by classifying triangles into layers [14]. Yang et al. [15] proposed a numerically stable solution to the point-in-polygon problem by combining the orientation method and the uniform subdivision technique. The criteria for determining whether a point lies inside a polygon according to the quasi-closest point was provided in their research. Liu et al. [16] proposed the relative closest triangle (RCT) method for locating points in 3D triangular meshes, and the method can even be applied in multi-materials. A method called PinMesh was recently proposed as a fast algorithm to perform exact 3D point location queries [13]. To compute the domain integrals in BEM by CDIM [5,6], a method based on the unit outward normal of the temporary boundary points near the query point is proposed to judge the location of the query point.

Most of the above algorithms are suitable for polygons or polyhedrons. In this paper, a different method based on the boundary integral equation (BIE) is proposed for bounded domains with arbitrary shapes in 2D and 3D. The method can also be implemented in a CAE model obtained after preprocessing by using traditional boundary elements/cells. In this case, the integrations are performed on each boundary element/cell. Thus, it can be easily applied in the finite element method (FEM) and boundary element method (BEM) [17–19]. In addition, by coupling with Non-uniform rational B-splines (NURBS) in isogeometric analysis (IGA), the method can be applied in CAD models without preprocessing [20]. This is because most CAD packages use boundary representation (B-rep) data to model the solids, and the boundary curves or surfaces are further represented by parameter functions. The main idea of IGA is to use the same parametric functions for geometric construction to approximate the fields in numerical analysis, and the

* Corresponding author at: State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China.
E-mail addresses: qiaowang@whu.edu.cn (Q. Wang), zw_mxx@whu.edu.cn (W. Zhou).

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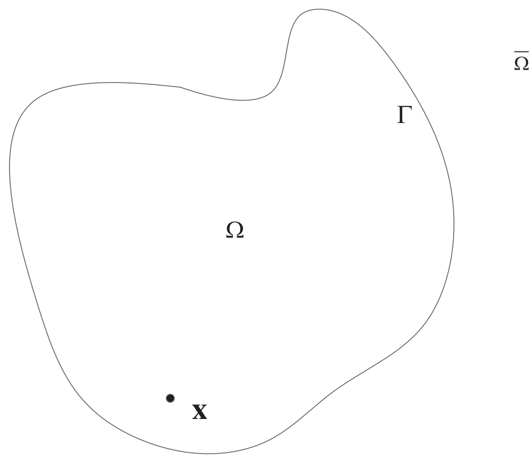

Fig. 1. Problem definition.

shape of the model can be reproduced exactly at all stages. However, in this paper, we only use parametric functions to reproduce the geometry, since no fields must be approximated. One advantage of coupling with isogeometric analysis is that the data used for geometry modeling in CAD can be used directly in the proposed method for the point location problem. NURBS are applied in most CAD packages, and the refinement can be done directly and easily by known algorithms or in CAD functions.

The proposed method is time-consuming for large-scale problems, since the time complexity is $O(NM)$, where N and M are the numbers of points and boundary integral cells, respectively. Fortunately, fast algorithms, such as the fast multipole method (FMM) [21] can easily be coupled with the BIE equation to reduce the time complexity to nearly $O(N+M)$. FMM has successfully been implemented in BEM [22,23] and the meshless method [24–28] as a fast solver, and it can also be applied for fast computation of the domain integrals [29,30]. The application of FMM in the proposed method can be easily implemented as in BEM and the main formulations are provided in this paper.

This paper is organized as follows. The boundary integral equation-based method for the point location problem is introduced in the second section. Discretization methods, including the traditional boundary element/cell method and the isogeometric cell method, are introduced in the third section, followed by the fast multipole method in the fourth section. Finally, examples are given in the fourth section to demonstrate the accuracy of the proposed method.

2. Boundary integral equation

In computational geometry, the point location problem is a fundamental problem: Given a subdivision of a solid, identifying which region of the subdivision contains a given test point. The solid usually subdivided by polygon or polyhedron. This problem can be equivalent to determine which point is in a given region. In this paper, the region can be a solid with arbitrary shape in CAD. Suppose Ω is a bounded domain with boundary Γ, $\overline{\Omega} = \mathbb{R}^d - \Omega - \Gamma$, and point $\mathbf{x} \in \mathbb{R}^d, d = 2, 3$, (see Fig. 1). The problem defined in this paper is giving the relationship between point **x** and Ω, i.e., **x** in Ω or not. A numerical method based on the BIE is used to solve this problem, and the BIE for the Laplace equation is introduced in this section.

The control equation for the Laplace equation is

$$\nabla^2 u = 0 \tag{1}$$

and the well-known BIE for Eq. (1) is [17]

$$c(\mathbf{x})u(\mathbf{x}) = \int_\Gamma u^*(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) - \int_\Gamma q^*(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \tag{2}$$

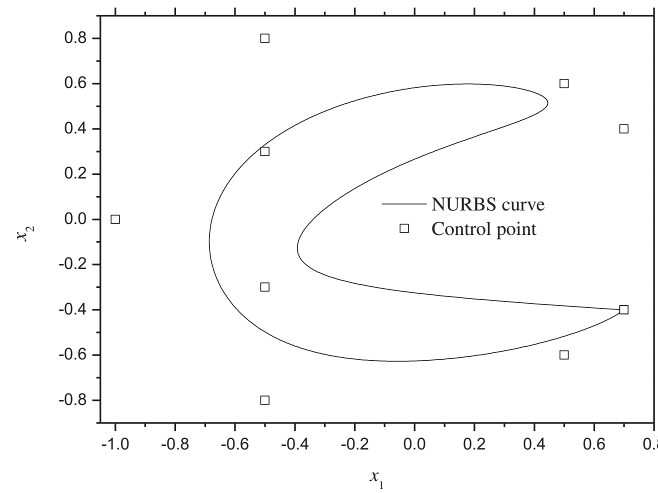

Fig. 2. A NURBS curve.

where $q(\mathbf{y}) = \partial u(\mathbf{y})/\partial \mathbf{n}(\mathbf{y})$ are the potential flows on the boundary point **y** and $\mathbf{n}(\mathbf{y})$ is the unit outward normal on the boundary point **y**. The $u^*(\mathbf{x}, \mathbf{y})$ are the fundamental solutions of the Laplace equation, which in the 2D case can be written as

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\ln\left(\frac{1}{r(\mathbf{x},\mathbf{y})}\right) \tag{3}$$

where $r(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is the distance between **x** and **y**.

In the 3D case, $u^*(\mathbf{x}, \mathbf{y})$ can be written as

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r(\mathbf{x},\mathbf{y})} \tag{4}$$

and

$$q^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \tag{5}$$

In Eq. (2), $c(\mathbf{x})$ is a coefficient related to the position of point **x**:

$$c(\mathbf{x}) = \begin{cases} 0, & \forall \mathbf{x} \in \bar{\Omega} \\ 1, & \forall \mathbf{x} \in \Omega \\ \frac{1}{2} & \mathbf{x} \text{ is on a smooth portion of } \Gamma \\ c_d, & \mathbf{x} \text{ is at a sharp corner on } \Gamma \end{cases} \tag{6}$$

where c_d is related to the space angle of the corner.

Eq. (2) is the basic BIE equation for the boundary element method (BEM), and one can solve it with given boundary conditions. In this paper, we need not solve this equation, and it is used to judge the location of point **x**.

To obtain the coefficient $c(\mathbf{x})$, one must know all the boundary values of u and q. A special boundary condition can be applied as

$$u(\mathbf{x}) = 1, q(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma \tag{7}$$

and the solution of Eq. (1) will be $u(\mathbf{x}) = 1$ for all points $\mathbf{x} \in \Omega$.

Under the boundary condition defined by Eq. (7), the corresponding solution $u(\mathbf{x}) = 1$ in Ω holds for an arbitrary bounded domain. This method, called the rigid body motion method, can be used to compute the diagonal elements in the coefficient matrix that contains the strongly singular integrals in BEM.

Eq. (2) can now be written as

$$c(\mathbf{x}) = -\int_\Gamma q^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \tag{8}$$

If **x** is on the boundary, Eq. (8) will have a strongly singular integral, and this is one of the most difficult problems in the BIE. However, Eq. (8) will not be used for the case when $\mathbf{x} \in \Gamma$. As discussed in the