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Improved Kansa RBF method for the solution of nonlinear boundary value problems

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ABSTRACT

We apply the Kansa–radial basis function (RBF) collocation method to two-dimensional nonlinear boundary value problems. In it, the solution is approximated by a linear combination of RBFs and the governing equation and boundary conditions are satisfied in a collocation sense at interior and boundary points, respectively. The nonlinear system of equations resulting from the Kansa–RBF discretization for the unknown coefficients in the RBF approximation is solved by directly applying a standard nonlinear solver. In a natural way, the value of the shape parameter in the RBFs employed in the approximation may be included in the unknowns to be determined. The numerical results of several examples are presented and analyzed.

1. Introduction

Radial basis function (RBF) methods have become popular in recent years in approximation theory as well as in the numerical solution of partial differential equations [1,4]. The most widely used RBF method for the latter class of problems is the RBF collocation method due to Kansa [13], known as the *Kansa method*. The popularity of this method is due to its meshlessness which means that only a set of points is required in the discretization of the continuous problem. This renders the implementation of the method particularly easy, especially for problems in complex geometries and/or in three dimensions. A disadvantage of the method is the (unknown) optimal choice of the shape parameter which is found in most RBFs. Various techniques have been proposed for the determination of an appropriate value of the shape parameter, see e.g. [5,14,15,18,20,24]. In addition, the RBF collocation methods discretization leads to highly ill-conditioned matrices and this has limited the accuracy to a certain level. Traditionally, RBF expansions have been augmented with linear combinations of low degree polynomial basis functions primarily for theoretical reasons. This approach, however, leads to little or no improvement in accuracy and has therefore been largely ignored in most applications. Recently, however, Yao et al. [30] discovered that the accuracy can be significantly improved if the RBF approximations are enriched with higher degree polynomial basis functions. Although, in general, high degree polynomials are numerically notoriously unstable, when coupled with RBFs, this instability is somehow tamed. From numerical observations, if the RBF expansion is augmented with low degree polynomial basis functions, the major contribution to the accuracy of the approximation is due to the RBFs. In contrast, when the expansion is augmented with higher degree polynomial basis functions, it is the polynomials that gradually take over as the major contributors to the accuracy and RBFs play the (minor) role of merely stabilizing the system. In this paper, we adopt this new modified RBF collocation approach of enriching the RBF expansions with polynomial basis functions to improve the accuracy. In [12] we presented some preliminary results for the solution of second order nonlinear boundary value problems using the Kansa method. In the method proposed in [12] the solution is expressed as a linear combination of only radial basis functions (see (2.2) below) without the inclusion of the polynomial basis as proposed in the current study. The latter *improved Kansa approximation* (see (2.5) below) yields considerably superior accuracy than the approach used in [12]. Furthermore, in this paper we also consider the case of using a predetermined shape parameter instead of determining it from the nonlinear solver.

In this work we shall consider the solution of boundary value problems for nonlinear partial differential equations. The RBF discretization of these problems invariably leads to systems of nonlinear equations which we shall solve using standard software. In particular, we shall be using the MATLAB© optimization toolbox routines `fsolve` and `lsqnonlin`. Since we shall be solving nonlinear problems it seems

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natural to include the (unknown) value of the shape parameter in the set of unknowns of the problem. Thus the solution of the nonlinear problem yields not only the coefficients in the RBF approximation to the solution but, also, an appropriate value of the shape parameter. As mentioned earlier, when the RBF expansions are augmented with high degree polynomial basis functions, the RBFs play a merely stabilizing (and minor) role with regard to accuracy and the shape parameter can be chosen more freely. This provides another alternative for choosing the shape parameter. Moreover, one of the main attractions of the proposed method is its simplicity and the ease with which it can be implemented.

The paper is organized as follows. In Section 2 we present the type of second order boundary value problems considered and describe the formulation of the Kansa method for their solution. The results of three numerical examples are presented in Section 3. In Section 4 we describe the formulation of the proposed method to fourth order boundary value problems and in Section 5 present the results of two further numerical examples. Finally, some concluding remarks are provided in Section 6.

2. The Kansa method

2.1. The problem

We consider the boundary value problem in $\mathbb{R}^2$

$$\mathcal{L}u = f \quad \text{in} \quad \Omega, \tag{2.1a}$$

subject to the boundary condition

$$\mathcal{B}u = g \quad \text{on} \quad \partial\Omega, \tag{2.1b}$$

where $\mathcal{L}$ is a second order nonlinear elliptic operator and $\mathcal{B}$ is a linear (or nonlinear) operator describing the boundary condition.

2.2. The method

In Kansa's method [13] we approximate the solution u of boundary value problem by a linear combination of RBFs

$$u_{\mathrm{N}}(x, y) = \sum_{\mathrm{n}=1}^{\mathrm{N}} a_{\mathrm{n}}\, \phi_{\mathrm{n}}(x, y), \qquad (x, y) \in \overline{\Omega}. \tag{2.2}$$

The RBFs $\phi_{\mathrm{n}}(x, y), \mathrm{n} = 1, \dots, \mathrm{N},$ can be expressed in the form

$$\phi_{\mathrm{n}}(x, y) = \Phi(r_{\mathrm{n}}), \quad \text{where} \quad r_{\mathrm{n}}^2 = (x - \mathrm{x_n})^2 + (y - \mathrm{y_n})^2. \tag{2.3}$$

Thus each RBF ϕ_{n} is associated with a point $(\mathrm{x_n}, \mathrm{y_n})$. These points $\{(\mathrm{x_n}, \mathrm{y_n})\}_{\mathrm{n}=1}^{\mathrm{N}}$ are usually referred to as *centers*. We shall assume that we have $\mathrm{N_{int}}$ interior centers $\{(\mathrm{x_n}, \mathrm{y_n})\}_{\mathrm{n}=1}^{\mathrm{N_{int}}}$ and $\mathrm{N_{bry}}$ boundary centers $\{(\mathrm{x_n}, \mathrm{y_n})\}_{\mathrm{n}=\mathrm{N_{int}}+1}^{\mathrm{N_{int}}+\mathrm{N_{bry}}}$. We take $\mathrm{N} = \mathrm{N_{int}} + \mathrm{N_{bry}}$.

In [29], for scattered data interpolation problems, expansions of certain types of conditionally positive definite RBFs were augmented by low degree polynomial basis functions to ensure the invertibility of the resultant matrices. In the case of the Kansa method, Hon and Schaback [11] indicated that the corresponding resultant matrices may be singular in only very rare cases. More recently, Fasshauer [8] proposed a modification of the Kansa method, based on Hermite collocation, to ensure the invertibility of the resulting coefficient matrix. However, because of the complexity of the implementation of RBF Hermite collocation, this approach has apparently not been well received and, despite the invertibility issue, the Kansa method remains very popular. Furthermore, in terms of accuracy there is no evidence of an obvious benefit in adding these terms and, therefore, in most applications they have been ignored. Recently, in [30] it was shown that the addition of higher degree polynomial basis functions in the method of particular solutions (MPS) lead to a significant improvement in accuracy. As will we described in the sequel, the same approach is also equally effective for the Kansa method.

Let $\mathbb{P}_{\mathrm{p}}$ be the set of bivariate polynomials of degree up to p and $\{p_{\mathrm{k}}\}_{\mathrm{k}=1}^{\mathrm{K}}$ be a basis of $\mathbb{P}_{\mathrm{p}}$ [11,30]. It is known that the number of polynomial terms for degree p is $\mathrm{K} = (\mathrm{p}+1)(\mathrm{p}+2)/2$. The polynomial basis is thus

$$p_{\mathrm{k}}(x, y) = x^{i-j} y^{j}, \quad 0 \le j \le i,\ 0 \le i \le \mathrm{p}, \quad \text{for} \quad \mathrm{k} = 1, \dots, \mathrm{K}. \tag{2.4}$$

In the modified Kansa method with an augmented polynomial basis, the approximation (2.2) of the solution of boundary value problem is thus replaced by

$$u_{\mathrm{N}}(x, y) = \sum_{\mathrm{n}=1}^{\mathrm{N}} a_{\mathrm{n}}\, \phi_{\mathrm{n}}(x, y) + \sum_{\mathrm{k}=1}^{\mathrm{K}} a_{\mathrm{N+k}}\, p_{\mathrm{k}}(x, y), \qquad (x, y) \in \overline{\Omega}. \tag{2.5}$$

An example of an RBF is the *normalized multiquadric* basis function (MQ)

$$\phi_{\mathrm{n}}(x, y) = \Phi(r_{\mathrm{n}}) = \sqrt{(c r_{\mathrm{n}})^2 + 1}, \tag{2.6}$$

where c is the *shape parameter*. Such shape parameters are often present in RBFs and the determination of their optimal value remains a major challenge.

Alternatively, one may use polyharmonic splines (PS) given by

$$\phi_{\mathrm{n}}(x, y) = \Phi(r_{\mathrm{n}}) = \begin{cases} r_{\mathrm{n}}^{2\ell-1}, & \text{in 3D,} \\ r_{\mathrm{n}}^{2\ell} \log r_{\mathrm{n}}, & \text{in 2D,} \end{cases} \qquad \ell = 1, 2, 3, \dots \tag{2.7}$$

The RBFs r^{2n-1} are also known as the radial power RBFs [9] and may be used for problems in all (both even and odd) dimensions. Clearly, the advantage of PS and radial power RBFs is the absence of a shape parameter.

In addition to the centers we consider the *collocation points* $\{(x_{\mathrm{m}}, y_{\mathrm{m}})\}_{\mathrm{m}=1}^{\mathrm{M}} \in \overline{\Omega}$. Of these, we have $\mathrm{M_{int}}$ interior collocation points $\{(x_{\mathrm{m}}, y_{\mathrm{m}})\}_{\mathrm{m}=1}^{\mathrm{M_{int}}}$ and $\mathrm{M_{bry}}$ boundary collocation points $\{(x_{\mathrm{m}}, y_{\mathrm{m}})\}_{\mathrm{m}=\mathrm{M_{int}}+1}^{\mathrm{M_{int}}+\mathrm{M_{bry}}}$. We take $\mathrm{M} = \mathrm{M_{int}} + \mathrm{M_{bry}}$.

Note that the number of centers is normally taken to be less than the number of collocation points.

The coefficients $\{a_{\mathrm{n}}\}_{\mathrm{n}=1}^{\mathrm{N+K}}$ in Eq. (2.5) are determined from the collocation equations

$$\mathcal{L}u_{\mathrm{N}}(x_{\mathrm{m}}, y_{\mathrm{m}}) = f(x_{\mathrm{m}}, y_{\mathrm{m}}), \quad \mathrm{m} = 1, \dots, \mathrm{M_{int}}, \tag{2.8a}$$

$$\mathcal{B}u_{\mathrm{N}}(x_{\mathrm{m}}, y_{\mathrm{m}}) = g(x_{\mathrm{m}}, y_{\mathrm{m}}), \quad \mathrm{m} = \mathrm{M_{int}} + 1, \dots, \mathrm{M_{int}} + \mathrm{M_{bry}}. \tag{2.8b}$$

In addition to (2.8a) and (2.8b) we impose the standard insolvency conditions [9, Chapter 6], see also [3],

$$\sum_{\mathrm{n}=1}^{\mathrm{N_{int}}} a_{\mathrm{n}} \{\mathcal{L}\, p_{\mathrm{k}}\}(\mathrm{x_n}, \mathrm{y_n}) = 0, \quad \mathrm{k} = 1, \dots, \mathrm{K}, \tag{2.8c}$$

and

$$\sum_{\mathrm{n}=\mathrm{N_{int}}+1}^{\mathrm{N}} a_{\mathrm{n}} \{\mathcal{B}\, p_{\mathrm{k}}\}(\mathrm{x_n}, \mathrm{y_n}) = 0, \quad \mathrm{k} = 1, \dots, \mathrm{K}. \tag{2.8d}$$

In fact we may combine (2.8c) and (2.8d) as

$$\sum_{\mathrm{n}=1}^{\mathrm{N_{int}}} a_{\mathrm{n}} \{\mathcal{L}\, p_{\mathrm{k}}\}(\mathrm{x_n}, \mathrm{y_n}) + \sum_{\mathrm{n}=\mathrm{N_{int}}+1}^{\mathrm{N}} a_{\mathrm{n}} \{\mathcal{B}\, p_{\mathrm{k}}\}(\mathrm{x_n}, \mathrm{y_n}) = 0, \quad \mathrm{k} = 1, \dots, \mathrm{K}. \tag{2.8e}$$

We have $\mathrm{M} + \mathrm{K}$ equations in $\mathrm{N} + \mathrm{K}$ unknown coefficients $\boldsymbol{a} = [a_1, a_2, \dots, a_{\mathrm{N+K}}]^T$ and we therefore take $\mathrm{M} \ge \mathrm{N}$.

In case the shape parameter is included in the unknowns then the number of unknowns becomes $\mathrm{N} + \mathrm{K} + 1$ consisting of $\boldsymbol{a}$ and c and we need to take $\mathrm{M} \ge \mathrm{N} + 1$.

Since the operator $\mathcal{L}$ is nonlinear, the system of $\mathrm{M} + \mathrm{K}$ equations (2.8a), (2.8b), and (2.8e) is nonlinear and can be written in the form