Engineering Analysis with Boundary Elements 000 (2017) 1–18

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A radial basis function method for fractional Darboux problems

Chandhini G. [a,*], Prashanthi K.S. [a], V. Antony Vijesh [b]

[a] Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal 575025, India
[b] School of Basic Sciences, Indian Institute of Technology Indore, Indore 452017, India

ARTICLE INFO

Keywords:
Fractional Darboux problem
Radial basis function
Collocation
Successive approximation
Gauss–Jacobi quadrature

ABSTRACT

In this paper, a radial basis function (RBF) collocation known as Kansa's method has been extended to solve fractional Darboux problems. The fractional derivatives are described in the Caputo sense. Integration of radial functions that appears due to fractional derivatives have been dealt using Gauss–Jacobi quadrature method. The equation has been linearized using successive approximation. A few test problems have been solved and compared with available solutions. The effect of RBF shape parameter on accuracy and convergence has also been discussed.

© 2017 Elsevier Ltd. All rights reserved.

1. Introduction

Darboux problems, where the governing equation is of hyperbolic in nature, in general arises in wave phenomena. Consider,

$$\begin{aligned} D_{xy} = \frac{\partial^2 u}{\partial x \partial y} &= f(x, y, u(x, y)), \quad (x, y) \in J \\ u(x, 0) &= g(x); \quad x \in [0, a] \\ u(0, y) &= h(y); \quad y \in [0, b] \end{aligned} \tag{1.1}$$

where $a, b > 0$, $J := [0, a] \times [0, b]$ and g and h are continuously differentiable functions.

Sometimes the Darboux problem is also referred as the Goursat problem. Certain classical problems of mathematical physics and rigid body dynamics are expressed in terms of Darboux problems. They can also be considered as a limiting case of tricomi problem. Efforts to solve (1.1) numerically is dated back to 1960s. These attempts are made by Day [1,2], Jain and Sharma [3] and Gourlay [4]. They are based on Trapezoidal or other quadrature formulae and Runge–Kutta type methods. Later a nonlinear trapezoidal formula based on geometric means [5] and harmonic means [6] are also considered in solving Goursat problems. In [7], a general class of difference schemes for this problem have been attempted. In 2011, the problem has been solved in a triangular domain with mixed Dirichlet and impedance boundary conditions [8], based on Runge–Kutta method and trapezoidal formula.

In recent years, fractional order differential equations (FDEs) are attracting not only mathematicians but also engineers and scientists from various fields. It is found that these equations can be used to model many natural phenomena and physical problems more accurately than their classical counterparts. To name a few, some of the problems where numerical schemes have been useful are: plasma transport problem with anomalous diffusion [9], fractional order Bloch equation that provides basis for nuclear magnetic resonance spectroscopy and magnetic resonance imaging [10] (NMR and MRI, respectively), description of the dynamic events that occur in biological tissues [11], fractional model for the shafting system of the water jet mixed-flow pump during the startup process [12], fractional neutron point kinetic model to analyse the dynamic behaviour of neutrons [13], etc.

In the present article, we have considered the fractional Darboux problem in the following form:

$$\begin{aligned} {}^cD^{\tilde{\alpha}}u(x, y) = \frac{\partial^{\alpha_1+\alpha_2}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}} &= f(x, y, u(x, y)), \quad (x, y) \in J, \\ u(x, 0) &= g(x) \quad x \in [0, a]; \\ u(0, y) &= h(y); \quad y \in [0, b], \end{aligned} \tag{1.2}$$

where $a, b > 0$, $J := (0, a] \times (0, b]$, $\tilde{\alpha} = (\alpha_1, \alpha_2) \in (0, 1] \times (0, 1]$ and g and h are continuously differentiable functions with $g(0) = h(0)$. The function $f : J \times \mathbb{R} \to \mathbb{R}$is a continuous function and satisfies Lipschitz condition with respect to the third variable u with Lipschitz constant L.

Researchers, namely, Abbas, Benchohra and Vityuk have worked extensively on the existence and uniqueness of various classes of fractional Darboux problem for hyperbolic type; see [14] and the references therein. The problems considered were fractional equations or inclusions with and without delay terms in various forms. Results are also established for equations that involves impulsive effect. Vityuk and Mykhailenko [15] have obtained the sufficient conditions of the existence and uniqueness of the solution of implicit fractional Darboux problem and also provided some numerical solutions.

Past two decades have witnessed an extensive development of RBF based schemes for solving PDEs. The reasons for such an interest towards

* Corresponding author.
E-mail addresses: chandhini@nitk.edu.in (C. G.), ksprashanthi@gmail.com (P. K.S.), vijesh@iiti.ac.in (V.A. Vijesh).

https://doi.org/10.1016/j.enganabound.2017.10.001
Received 29 March 2017; Received in revised form 6 August 2017; Accepted 2 October 2017
Available online xxx
0955-7997/© 2017 Elsevier Ltd. All rights reserved.

RBFs are that they are naturally grid free and depends only on the distances which makes application to higher dimensional problems straight forward. Observing the advantages of RBFs in multivariate approximation [16], Kansa suggested [17] an RBF based collocation method for both boundary and initial boundary value problems. Following this, various other formulations based on radial functions have also been proposed [18–20] and applied to many application problems [21–23].

In the context of the solution of the equations involving fractional derivatives, a few attempts have been made to use radial functions in the discretization schemes. RBF based collocations (both in global/local form) have been applied to some of the important fractional partial differential differential equations such as anomalous subdiffusion, fractional diffusion and advection–diffusion problems [24–28]. Also other equations for which radial functions have been considered are fractional variants of Schrödinger equation [29] and telegraph equation [30], Sine–Gordon and Klein–Gordon equations [31], to name a few. In another novel work [32], a scheme called Laplace transformed boundary particle method (LTBPM) was proposed, which simulate long time-history fractional diffusion systems more effectively than the finite difference discretization. However, these problems consist of time-fractional PDEs, where RBFs have been chosen only for discretizing the terms involving integer-order space derivatives. In [33], the authors have solved 1-D space fractional diffusion problem using radial functions based RBF-QR algorithm. They dealt with the integral of radial functions by analytically integrating the truncated Maclaurin series of GA RBF. Pang et al. [34] solved generalized space-fractional advection–dispersion equation using RBF collocation, while numerically integrating radial functions using Gauss–Jacobi quadrature formula.

The present article focuses on extending RBF based collocation scheme to fractional Darboux problem defined in (1.2), which involves mixed fractional partial derivatives. Even for integer order Darboux problems, there are only a few attempts to solve them numerically and no efforts are made based on RBF based schemes. Hence, our main contribution is in extending the collocation scheme to both integer and fractional order Darboux problem. Section 2 provides some of the basic definitions on mixed fractional derivatives and integrals and also a result that ensures the existence of the solution and convergence of successive approximation to the solution of (1.2). Derivation of Kansa's collocation [17] for fractional Darboux equations are detailed in Section 3. For some of the radial functions, strategies such as optimization of shape parameter and variable shape parameters have also been employed. The scheme then tested on variety of example problems and these results are analysed in Section 4. Finally, the article is concluded by summarising both advantages and disadvantages with some suggestions for possible future improvements.

2. Basic definitions and results

In this section, we introduce the notations, definitions and some of the fundamental assumptions that are considered in the present work.

2.1. Fractional integrals and derivatives

Unlike in the case of integer order, fractional order derivative has several definitions. We have considered Riemann–Liouville and Caputo definitions for fractional integrals and derivatives, respectively.

Definition 2.1. The Riemann–Liouville fractional integrals of order $\alpha \in (0, \infty)$ of a function $f \in L^1([0, a])$; $a > 0$ is defined by

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds,$$

for $x \in [0, a]$, where $\Gamma(.)$ is Euler's gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,\ \alpha > 0$.

Definition 2.2. The Caputo derivative of order $\alpha \in (0, 1]$ of a function f, where $f' \in L^1([0, a])$ is defined by

$$^c D_0^\alpha f(x) = I_0^{1-\alpha} f'(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s)ds; \tag{2.3}$$

for $x \in [0, a]$.

Since we are considering fractional order Darboux problem as in Eq. (1.2), which requires the definition of mixed (partial) fractional derivative, the extension of the Caputo definition to partial fractional derivatives are considered in the following [35].

Definition 2.3. The Riemann–Liouville fractional integral of order $\alpha \in (0, \infty)$ of a function $u \in L^1(J)$ with respect to x is defined by

$$I_{0,x}^\alpha u(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s, y)ds,$$

for $x \in [0, a]$and $y \in [0, b]$. Analogously, $I_{0,y}^\alpha u(x, y)$ can also be defined.

Definition 2.4. The Caputo fractional derivative of order $\alpha \in (0, 1]$ of a function u, where $\frac{\partial u}{\partial x} \in L^1(J)$, with respect to x is defined by

$$^c D_{0,x}^\alpha u(x, y) = I_{0,x}^{1-\alpha} \frac{\partial}{\partial x} u(x, y),$$

for $x \in [0, a]$ and $y \in [0, b]$. Analogously, $^c D_{0,y}^\alpha u(x, y)$ can also be defined.

Definition 2.5. Let $\bar{\alpha} = (\alpha_1, \alpha_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J)$. The mixed Riemann–Liouville integral of order $\bar{\alpha}$ of u is defined by

$$I_\theta^{\bar{\alpha}} u(x, y) = I_0^{\alpha_1} I_0^{\alpha_2} u(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} u(s, t)dtds$$

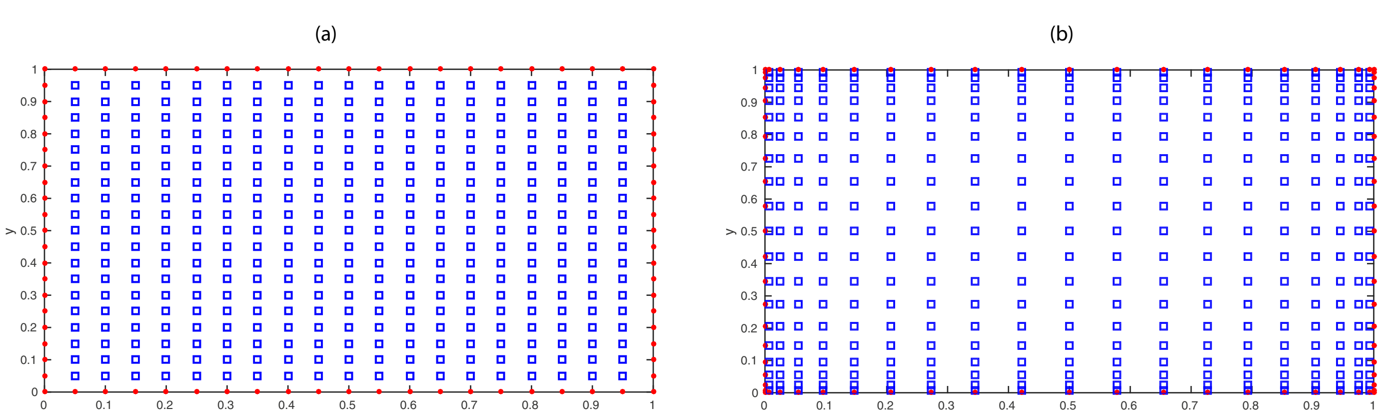

Fig. 1. Schematic of nodal distributions. (a) uniform, (b) nonuniform.

Download English Version:

https://daneshyari.com/en/article/6925084

Download Persian Version:

https://daneshyari.com/article/6925084

Daneshyari.com