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A homogenization boundary function method for determining inaccessible boundary of a rigid inclusion for the Poisson equation

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ABSTRACT

In this paper, the problem for determining the inner boundary of the Poisson equation in an arbitrary doubly-connected plane domain is solved, which recovers an unknown inner boundary of a rigid inclusion under the over-specified Cauchy data on the accessible outer boundary. First, a homogenization function is derived to annihilate the Dirichlet and Neumann data over-specified on the outer boundary. Second, a new concept of boundary functions is introduced, which automatically satisfy the homogeneous boundary conditions on the outer boundary. Besides the lowest order elementary boundary function, other higher-order boundary functions are obtained by multiplying the elementary boundary function to the Pascal triangle. Then, by a homogenization technique we can obtain a transformed Poisson equation in a reduced doubly-connected domain in terms of the transformed variable and solve it by the domain type collocation method, whose numerical solution is expanded by a sequence of boundary functions. The nonlinear equation for determining the unknown inner boundary is derived, which is convergent fast. The accuracy and robustness of present homogenization boundary function method are assessed through five numerical examples, by comparing the exact inner boundary to the recovered one under a large noisy disturbance.

1. Introduction

For the elliptic type partial differential equations (PDEs), there are different inverse problems. Among them the inverse geometry problems (IGPs) are of the most challenged ones because the solutions of IGPs depend nonlinearly on the boundary curve. Some typical examples of the IGPs could be the determination of the interface between liquid and solid phases and the detection of cracks and voids inside the solid materials.

The non-destructive testing is a popular method used in the engineering industry to detect the void and rigid inclusion. The problem is to detect the boundary of a void and rigid inclusion by means of the current flux and potential measurements on an accessible outer boundary in a doubly-connected domain. It is known that the IGPs are highly ill-posed [1–5], which inherit the non-characteristic property of the Cauchy problem for the elliptic type PDEs. Usually, a small error in the given data can terribly detract the accuracy of numerical solution in the prediction of inner shape. The regularized Trefftz method has been successfully applied to detect the different inner shapes in the two-dimensional voids [6].

For the related works on the IGPs in the steady-state heat conduction equation, the readers can refer [7–9]. The numerical methods used to reconstruct the voids in steady-state or transient heat conduction problems are numerous [10–15]. We study a rigid inclusion problem inside an arbitrary plane domain, of which Bormann et al. [16] and Ivanyshyn and Kress [17] have employed the method of fundamental solutions (MFS)-based reconstruction technique to detect the rigid inclusions for the two-dimensional stationary heat conduction equation of the isotropic media.

The remainder of this paper is arranged as follows. In Section 2 we specify the inner boundary determination problem of the Poisson equation in a doubly-connected plane domain and derive a homogenization function, which annihilates the over-specified Cauchy data on the outer boundary. Then, a homogenization technique and a new concept of boundary functions are introduced in Section 3, where in a new Poisson equation we express the transformed variable in terms of boundary functions as the bases, which satisfy the homogeneous boundary conditions on the outer boundary automatically. In Section 4 we describe the numerical algorithm of the homogenization/boundary function method for arbitrary prescribed source term in the Poisson equation. Five numerical examples are given in Section 5 to assess the capability of the

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new method in the detection of different rigid inclusions of solid materials. Finally, the conclusions are drawn in Section 6.

2. Homogenization function and variable transformation

2.1. Problem statement

We consider the following boundary determination problem of the Poisson equation:

$$\Delta u = f(x, y), \quad (x, y) \in \Omega, \tag{1}$$

$$u = h_1(x, y), \quad (x, y) \in \Gamma_1, \tag{2}$$

$$u_n = g(x, y), \quad (x, y) \in \Gamma_1, \tag{3}$$

$$u = h_2(x, y), \quad (x, y) \in \Gamma_2, \tag{4}$$

where $f(x, y)$, $h_1(x, y)$, $g(x, y)$ and $h_2(x, y)$ are given functions. Ω is a star-like doubly-connected plane domain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$. While Γ_1 denotes an outer boundary, Γ_2 is an inner boundary. On Γ_1 the Cauchy data are over-specified in order to determine the unknown inner boundary Γ_2. In the rigid inclusion detection problem one may set $h_2(x, y) = 0$. Notice that if the inclusion is an insulator, i.e. zero conductivity, then the condition $u = 0$ should be replaced by $u_n = 0$ on Γ_2. The one with $u_n = 0$ on Γ_2 to detect the cavity is more difficult to solve, which will be studied in other place.

The above n is an outward unit normal on Γ_1. If the inner boundary shape Γ_2 can be made available, together with the prescribed data $h_2(x, y)$, then the data are completed on the whole boundary, and the solution of the Poisson equation in the doubly-connected domain can be obtained. So the present inverse geometry problem (IGP) is such an inverse problem that giving over-specified data on the accessible outer boundary Γ_1, we seek an unknown inaccessible inner boundary Γ_2, where the Dirichlet data $h_2(x, y)$, $(x, y) \in \Gamma_2$ are given.

For the direct problem in a singly connected domain, giving conditions in Eqs. (2) and (3) are contradictory. However, because the presented IGP is in a doubly connected domain to seek an unknown inner boundary, giving conditions in Eqs. (2) and (3) are not contradictory, which are over-specified to help the solution of finding an unknown inner boundary. In fact, it includes a Cauchy problem in the doubly connected domain.

The outer boundary Γ_1 in the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ is described by $\Gamma_1 = \{(r, \theta) | r = \rho(\theta),\ 0 \le \theta \le 2\pi\}$, where $\rho(\theta)$ is a radius function of the outer boundary. In above, $u_n(\rho, \theta)$ is defined by [18]

$$u_n(\rho, \theta) = \eta(\theta)\left[\frac{\partial u(\rho, \theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho, \theta)}{\partial \theta}\right], \tag{5}$$

where

$$\frac{\partial u(\rho, \theta)}{\partial \rho} = \left.\frac{\partial u(r, \theta)}{\partial r}\right|_{r=\rho(\theta)},$$
$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta) + \rho'(\theta)^2}}, \tag{6}$$

in which the prime denotes the differential with respect to θ.

Without frequently switching the notations between the two different coordinates systems, we sometimes write function in terms of (x, y) rather than that in terms of (r, θ). Because the data $h_1(x, y)$ and $g(x, y)$ on the outer boundary Γ_1 are given, one has

$$h_1(\theta) := h_1(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta), \quad 0 \le \theta \le 2\pi, \tag{7}$$

$$g(\theta) = g(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta), \quad 0 \le \theta \le 2\pi. \tag{8}$$

For saving notations we use the same symbols h_1 and g to denote $h_1(x, y)$ and $g(x, y)$ on the outer boundary Γ_1.

2.2. Variable transformation

Our goal is converted the IGP stated in the previous section to solving a direct problem for the new variable in a new Poisson equation by using the domain type collocation method. The strategy is that we seek a variable transformation by

$$v(x, y) = u(x, y) - B^0(r, \theta), \tag{9}$$

and solve $v(x, y)$ by a new Poisson equation:

$$\Delta v = f(x, y) - \Delta B^0, \tag{10}$$

$$v(x, y) = 0, \quad v_n(x, y) = 0, \quad (x, y) \in \Gamma_1. \tag{11}$$

$B^0(r, \theta)$ plays a key role to diminish the over-specified boundary conditions on Γ_1, which will be searched in the next section. Because the boundary conditions on Γ_1 are homogeneous for the new Poisson equation, it is easily to find some suitable homogeneous bases for the new variable v, such that we can use the domain type collocation method to directly solve the above direct problem to find v, and then follows Eq. (9) one can easily obtain $u(x, y) = v(x, y) + B^0(r, \theta)$.

2.3. Homogenization function

In this section, we construct the homogenization function $B^0(r, \theta)$, which is a key point for the domain type method to solve $v(x, y)$.

Definition 1. A function defined in a domain Ω with Γ_1 being its outer boundary is said to be a *homogenization function*, if the over-specified Cauchy data on Γ_1 is satisfied by that function.

Now we propose a problem that does there exist a function $B^0(r, \theta)$ in the domain Ω, such that on the outer boundary $r = \rho(\theta)$ we have

$$B^0(\rho, \theta) = h_1(\theta), \tag{12}$$

$$B^0_n(\rho, \theta) = g(\theta), \tag{13}$$

where according to the Definition (5) of the normal derivative on Γ_1 and replacing $u(\rho, \theta)$ by $B^0(\rho, \theta)$ we have

$$\begin{aligned} B^0_n(\rho, \theta) &= \eta(\theta)\left[\left.\frac{\partial B^0(r, \theta)}{\partial r}\right|_{r=\rho(\theta)} - \frac{\rho'}{\rho^2}\left.\frac{\partial B^0(r, \theta)}{\partial \theta}\right|_{r=\rho(\theta)}\right] \\ &= \eta(\theta)\left[\frac{\partial B^0(\rho, \theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial B^0(\rho, \theta)}{\partial \theta}\right]. \end{aligned} \tag{14}$$

We can derive the following important result.

Theorem 1. *The homogenization function in terms of polar coordinates (r, θ):*

$$B^0(r, \theta) = h_1(\theta) + [r - \rho(\theta)]\frac{\partial u(\rho, \theta)}{\partial \rho} \tag{15}$$

in the domain Ω is a solution of Eqs. (12) and (13), i.e.,

$$B^0(\rho, \theta) = h_1(\theta), \quad B^0_n(\rho, \theta) = g(\theta).$$

Moreover, on the outer boundary Γ_1:

$$\frac{\partial B^0(\rho, \theta)}{\partial \rho} = \frac{\partial u(\rho, \theta)}{\partial \rho}, \tag{16}$$

$$\frac{\partial B^0(\rho, \theta)}{\partial \theta} = \frac{\partial u(\rho, \theta)}{\partial \theta}. \tag{17}$$

Proof. From Eq. (15) it is obvious that $B^0(\rho, \theta) = h_1(\theta)$ as specified by Eq. (12). In order to prove that $B^0(r, \theta)$ satisfies Eq. (13), proving Eqs. (16) and (17) is sufficient.