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Numerical solution of fractional telegraph equation by using radial basis functions

Vahid Reza Hosseini, Wen Chen, Zakieh Avazzadeh *

State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, College of Mechanics and Materials, Hohai University, Nanjing 210098, China

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ABSTRACT

In this paper, we implement the radial basis functions for solving a classical type of time-fractional telegraph equation defined by Caputo sense for $(1<\alpha\le 2)$. The presented method which is coupled of the radial basis functions and finite difference scheme achieves the semi-discrete solution. We investigate the stability, convergence and theoretical analysis of the scheme which verify the validity of the proposed method. Numerical results show the simplicity and accuracy of the presented method.

1. Introduction

In recent years fractional calculus has been implemented to describe some phenomena in physics and engineering. Also, fractional integral and derivative have been successful to describe many events in fluid mechanics, viscoelasticity, chemical physics, electricity, finance, control theory, biomedical engineering, heat conduction, diffusion problems and other sciences [13,22,25,28]. Fractional partial differential equations (FPDEs), particularly space- and time-fractional equations, have been widely studied to construct the existence of solution and validity of these problems [15,34,31,36]. In addition, finding the reliable and powerful numerical and analytical methods for solving FPDEs has been focused in the last two decades. According to the mathematical literature, fractional partial differential equations have been progressed in various problems in science and engineering such as the Schröinger, diffusion and telegraph fractional equations [4,14,15,17,23,35].

After the appearance of the remarkable work due to Orsinger and Zhao by using the Fourier technique [26], the fractional telegraph equation has been solved in several ways including Adomian decomposition method (ADM) [24], He's variational iteration method [8], He's homotopy perturbation method (HPM) [33], homotopy analysis method (HAM) [7] and other new methods [12].

In 2009, Wen et al. were the pioneers in using the Kansa method for solving the fractional diffusion equation [5]. After that the method was interested for solving the other fractional [11,23,27]. In this study, we implement the meshless method for solving the time-fractional telegraph equation by using a radial basis function (RBF). The presented method is coupled of the radial basis functions and finite difference scheme as is handled in [1,2,5,9,10].

The paper is organized in the following way. In Section 2, the Caputo fractional derivative and radial basis functions as the tools for performing the proposed method are described. In Section 3, firstly the discretization process of the problem in the t-direction via the finite difference scheme is described. Also, we will explain how to achieve the approximated solution by using the radial basis functions. Error analysis, stability and convergence of proposed method are discussed in Section 4. In Section 5, some numerical examples are demonstrated which confirm the accuracy and applicability of the method. The last section includes some other features of the presented method, conclusion and further ideas for future work.

2. Basic definitions

2.1. Fractional derivative

Definition. The Caputo fractional derivative operator of order $\alpha\ge 0$, of a function $F(x)$ is defined as

$$D_*^{\alpha}F(x)=\begin{cases}\dfrac{1}{\Gamma(k-\alpha)}\int_0^x (x-\xi)^{k-\alpha-1}F^{(k)}(\xi)\,d\xi, & k-1<\alpha<k,\ \ x>0,\\ F^{(k)}(x), & \alpha=k.\end{cases} \tag{1}$$

More properties of the fractional Caputo derivative can be found in [13,28]. Also, for further information about fractional calculus and

* Corresponding author. Tel.: +86 15240212636.
E-mail addresses: vrhosseini@gmail.com (V.R. Hosseini), chenwen@hhu.edu.cn (W. Chen), z.avazzadeh@gmail.com, z.avazzadeh@yahoo.com (Z. Avazzadeh).

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another definitions of fractional derivatives, one can consult the mentioned references.

2.2. Radial basis functions

Considering a finite set of interpolation points $\mathcal{X}=\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}\subset\mathbb{R}^d$ and a function $u:\mathcal{X}\to\mathbb{R}$, according to the process of interpolation using radial basis functions [3], the interpolant of u is constructed in the following form

$$(Su)(\mathbf{x})=\sum_{i=1}^{N}\lambda_i\phi(\|\mathbf{x}-\mathbf{x}_i\|)+p(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^d \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm and $\phi(\|\cdot\|)$ is a radial function. Also, $p(\mathbf{x})$ is a linear combination of polynomials on $\mathbb{R}^d$ of total degree at most $m-1$ as follows:

$$p(\mathbf{x})=\sum_{j=N+1}^{N+l}\lambda_j q_j(\mathbf{x}),\quad l=\binom{m+d-1}{d}. \tag{3}$$

Moreover, the interpolant Su and additional conditions must be determined to satisfy the system

$$\begin{cases}(Su)(\mathbf{x}_i)=u(\mathbf{x}_i), & i=1,2,\ldots,N,\\ \sum_{i=1}^{N}\lambda_i q_j(\mathbf{x}_i)=0 & \text{for all } q_j\in\Pi_{m-1}^d,\end{cases} \tag{4}$$

where Π_{m-1}^d denotes the space of all polynomials on $\mathbb{R}^d$ of total degree at most $m-1$. Now we have a unique interpolant (Su) of u if $\phi(r)$ is a conditionally positive definite radial basis function of order m [21]. The interested reader can see [3,6,18–20,29]. We will use the generalized thin plate splines (GTPS) which have the following form:

$$\phi(\|\mathbf{x}-\mathbf{x}_i\|)=\phi(r_i)=r_i^{2m}\log(r_i),\quad i=1,2,3,\ldots,\quad m=1,2,3,\ldots, \tag{5}$$

where $r_i=\|\mathbf{x}-\mathbf{x}_i\|$. We note that ϕ in Eq. (5) is $\mathbf{C}^{2m-1}$ continuous. Therefore, the higher order of partial differentials needs the higher order of thin plate splines. As $u(\mathbf{x})$ can be approximated by

$$u(\mathbf{x})\simeq\sum_{\mathbf{x}_i\in\mathcal{X}}\lambda_i\phi(\|\mathbf{x}-\mathbf{x}_i\|)+p(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^d, \tag{6}$$

for any partial differential operator $\mathcal{L}$, $\mathcal{L}u$ can be represented by

$$\mathcal{L}u(\mathbf{x})=\sum_{\mathbf{x}_i\in\mathcal{X}}\lambda_i\mathcal{L}\phi(\|\mathbf{x}-\mathbf{x}_i\|)+\mathcal{L}p(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^d. \tag{7}$$

The coefficients λ_i will be obtained by solving the system of linear equations.

3. Description of the method

Consider the following time-fractional telegraph equation of order $\alpha(1<\alpha\le 2)$:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\gamma_1\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}+\gamma_2 u(x,t)=\gamma_3\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\quad a\le x\le b,\quad 0\le t\le T \tag{8}$$

with the initial conditions

$$u(x,0)=g_1(x),\quad a\le x\le b \tag{9}$$

$$u_t(x,0)=g_2(x),\quad a\le x\le b, \tag{10}$$

and the boundary conditions

$$u(a,t)=h_1(t),\quad u(b,t)=h_2(t),\quad t\ge 0, \tag{11}$$

where $a,b,\alpha,g_1(x),g_2(x),h_1(t)$ and $h_2(t)$ are given and $\partial^\alpha u(x,t)/\partial t^\alpha$ represents the Caputo fractional derivative. Also, γ_1,γ_2 and γ_3 are constant given the coefficients.

According to Eq. (1), $\partial^\alpha u(x,t)/\partial t^\alpha$ can be written as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(2-\alpha)}\displaystyle\int_0^t\frac{\partial^2 u(x,\xi)}{\partial\xi^2}(t-\xi)^{1-\alpha}\,d\xi, & 1<\alpha<2,\\ \dfrac{\partial^2 u(x,\xi)}{\partial\xi^2}, & \alpha=2.\end{cases} \tag{12}$$

and regarding $(0<\alpha-1<1)$, we have

$$\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}=\begin{cases}\dfrac{1}{\Gamma(2-\alpha)}\displaystyle\int_0^t\frac{\partial u(x,\xi)}{\partial\xi}(t-\xi)^{1-\alpha}\,d\xi, & 1<\alpha<2,\\ \dfrac{\partial u(x,\xi)}{\partial\xi}, & \alpha=2.\end{cases} \tag{13}$$

In order to discretize the problem for $(1<\alpha<2)$ in the time direction, we substitute t^{n+1} into Eqs. (12) and (13), then the integrals can be partitioned as

$$\begin{aligned}\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha}&=\frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}}\frac{\partial^2 u(x,\xi)}{\partial\xi^2}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\\&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\int_{t^k}^{t^{k+1}}\frac{\partial^2 u(x,\xi)}{\partial\xi^2}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\frac{\partial^{\alpha-1}u(x,t^{n+1})}{\partial t^{\alpha-1}}&=\frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}}\frac{\partial u(x,\xi)}{\partial\xi}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\\&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\int_{t^k}^{t^{k+1}}\frac{\partial u(x,\xi)}{\partial\xi}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\end{aligned} \tag{15}$$

where $t^0=0$, $t^{n+1}=t^n+\delta t$, $n=0,1,2,\ldots,M$. Also, n can be increased to the time length with δt as the time step which $\delta tM=T$.

Approximations of the first and second order derivatives due to the forward finite difference formulae are defined as

$$\frac{\partial^2 u(x,\sigma)}{\partial t^2}=\frac{u(x,t^{n+1})-2u(x,t^n)+u(x,t^{n-1})}{\delta t^2}+o(\delta t^2), \tag{16}$$

$$\frac{\partial u(x,\sigma)}{\partial t}=\frac{u(x,t^{n+1})-u(x,t^n)}{\delta t}+o(\delta t), \tag{17}$$

where $\sigma\in[t^n,t^{n+1}]$. Replacement of Eqs. (16) and (17) into Eqs. (14) and (15), respectively, gives

$$\begin{aligned}\frac{\partial^\alpha u(x,t^{n+1})}{\partial t^\alpha}&=\frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}}\frac{\partial^2 u(x,\xi)}{\partial\xi^2}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\\&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\frac{u^{k+1}-2u^k+u^{k-1}}{\delta t^2}\\&\quad\times\int_{t^k}^{t^{k+1}}(t^{n+1}-\xi)^{\alpha-1}\,d\xi,\end{aligned} \tag{18}$$

and

$$\begin{aligned}\frac{\partial^{\alpha-1}u(x,t^{n+1})}{\partial t^{\alpha-1}}&=\frac{1}{\Gamma(2-\alpha)}\int_0^{t^{n+1}}\frac{\partial u(x,\xi)}{\partial\xi}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\\&=\frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n}\frac{u^{k+1}-u^k}{\delta t}\int_{t^k}^{t^{k+1}}(t^{n+1}-\xi)^{1-\alpha}\,d\xi,\end{aligned} \tag{19}$$

where $u^k=u(x,t^k)$, $k=0,1,\ldots,M$.

By considering $t^{n+1}-\xi=r$, the integral is easily obtained as

$$\begin{aligned}\int_{t^k}^{t^{k+1}}(t^{n+1}-\xi)^{\alpha-1}\,d\xi&=\frac{-1}{(2-\alpha)}r^{2-\alpha}\Big|_{t^{n-k+1}}^{t^{n-k}}\\&=\frac{1}{(2-\alpha)}\delta t^{2-\alpha}[(n-k+1)^{2-\alpha}-(n-k)^{2-\alpha}].\end{aligned} \tag{20}$$