Engineering Analysis with Boundary Elements 38 (2014) 40–48

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

On the solution of exterior plane problems by the boundary element method: A physical point of view

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ARTICLE INFO

Article history:
Received 12 March 2012
Accepted 8 October 2013
Available online 5 November 2013

Keywords:
Elasticity
Laplace equation
Boundary element method
Plane problems
Green's function
Exterior problem
Impedance matrix

ABSTRACT

The paper is devoted to the solution of Laplace equation by the boundary element method. The coupling between a finite element solution inside a bounded domain and a boundary integral formulation for an exterior infinite domain can be performed by producing a “stiffness” or “impedance matrix”. It is shown in a first step that the use of classical Green's functions for plane domains can lead to impedance matrices which are not satisfying, being singular or not positive-definite. Avoiding the degenerate scale problem is classically overcome by adding to Green's function a constant which is large compared to the size of the domain. However, it is shown that this constant affects the solution of exterior problems in the case of non-null resultant of the normal gradient at the boundary. It becomes therefore important to define this constant related to a characteristic length introduced into Green's function. Using a “‘slender body theory” allows to show that for long cylindrical domains with a given cross section, the characteristic length is shown as being asymptotically equal to the length of the cylindrical domain. Comparing numerical or analytical 3D and 2D solutions on circular cylindrical domains confirms this result for circular cylinders.

1. Introduction

The advantage of the boundary element method, compared to other approximate solutions is the most obvious when the method is used for problems on unbounded domains, either for solving completely linear systems of partial differential equations or very often as a complement of the finite element method in the case of problems containing local non-linearities.

It is well known that, in the case of 2D problems, the fundamental solution tends to infinity when the distance between source and observation point tends to infinity, either for the case of Laplace equation or for the elasticity operator. A consequence is that if the resultant of sources is not null, the solution does not vanish at infinity. A main point of interest in the literature is the well-posedness of the exterior boundary value problem. Due to the logarithmic behaviour of the fundamental solution at infinity, the well-posedness has been studied primarily in the case where the resultant of sources (or forces in problems related to elasticity equations) on the finite boundary is null [1–7]. However many practical problems need the application of sources or forces whose resultant is non-null; this case has been considered in [8,9]. The validness of the corresponding integral equation has been proved in detail by [10] for the standard integral equation using Somigliana equality (with a new kernel) and by [11] for the regularized integral equation without restriction on the kernel which is used. So, in this paper, we shall assume that the consideration of an exterior problem by standard BEM is licit even if the resultant of sources or forces applied at the boundary is non-null.

A second difficulty related to 2D problems is the loss of uniqueness of the solution when the domain under consideration has specific dimensions and when the classical logarithmic function for an infinite domain is used. It was early recognized [12] and the usual practical way to circumvent this problem is to add a constant to Green's function which must be adjusted to the dimensions of the domain [13,9]. This method leads however to the obtaining of the potential up to an additional arbitrary constant value, this method being related to a convenient “scaling” of the distances introduced when using the fundamental solution [14]. Other methods are used to perform a “regularization” of the problem and to recover the uniqueness [15–20]. This problem is recurrent within the literature [21,22,7,23,24]. The link between the condition number of the BEM matrices and the scaling of the problem has been also investigated [25]. Two points remain unclear in this context. First, it is clear that

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http://dx.doi.org/10.1016/j.enganabound.2013.10.005

some constants added to the fundamental solution must be avoided, leading to non-uniqueness of the solution. However, it is not always clear if all other values can still be used. Indeed, it is practically always considered that the constant is arbitrary and that all solutions related to different values of the arbitrary constant are the same, up to an additional constant. This paper addresses these two points in the case of Laplace equation. The purpose of the present paper is therefore

- to present in Section 2 an example where the use of the classical singular solution for Laplace equation within the usual formulation of the boundary integral equation leads to non-physical solutions. The example recovers the loss of uniqueness for some specific problems and shows that in addition to the loss of uniqueness, some fundamental solutions can lead to a loss of positiveness of the "impedance matrix" induced by the boundary element formulation;
- to prove next that for interior problems any constant added to the fundamental solution does not change the solution which is given up to an added constant and that in the case of exterior problems, solutions related to different constants added to the fundamental solution are different for Dirichlet or mixed value problems. Therefore, it is clearly of importance to find the value of the "right" constant related to a given physical problem or the right "characteristic length" associated to this problem;
- to show in Section 3 by using the "slender body theory" that the solution to a given 2D boundary value problem can be obtained as the asymptotic limit of the solution to an associate 3D problem over a long cylinder as soon as the associated "characteristic length" is equal to the length of the cylinder;
- to confirm in Section 6 the result of the "slender body theory" by comparing the results coming from 3D problems over long cylinders, these results being obtained by various analytical (Section 4) or numerical (Section 5) solutions.

2. Discussion of the direct formulation of the boundary element method in the case of the exterior problem for Laplace equation

2.1. Example of loss of uniqueness and appearance of unphysical results on a numerical solution built from the classical formulation of the exterior problem

Our discussion will start with the classical formulation of the exterior "Dirichlet" Boundary Value Problem. Let us therefore consider the solution u of the exterior problem of Laplace equation on a plane domain D having a boundary ∂D for Dirichlet boundary conditions. The classical discretized integral equation writes (see Appendix A)

$$[H][u]=[G][q] \tag{1}$$

where $[u]$ and $[q]$ contain the nodal displacements and the nodal values of the normal flux, respectively, while $[H]$ and $[G]$ are built from the interpolation functions, the geometry of the elements and from Green's function (and its derivatives).

Green's function which is used at the beginning is the classical expression for 2D problems given by

$$G=\frac{1}{2\pi}\ln(1/r) \tag{2}$$

r being the distance between source and observation point.

The physical soundness of the results will be studied by using the eigenvalues of

- the matrix $[G]$,
- the symmetric part $[K]$ of the "impedance matrix" $[K_1]$ (cf. Appendix A) which allows the computation of the supply of energy from

  $$W=\tfrac{1}{2}[u]^T[F]=\tfrac{1}{2}[u]^T[K_1][u]=\tfrac{1}{2}[u]^T[K][u] \tag{3}$$

  where $[K_1]$ is the "impedance matrix" which is built by using matrices $[H]$ and $[G]$ as in [26]:

  $$[K_1]=[Q][G]^{-1}[H] \tag{4}$$

  where $[Q]$ is a matrix allowing the condensation of boundary stresses on nodal forces.

It may be noticed that the "impedance matrix" $[K_1]$ is in general not symmetric, due to the fact that $[G]$ and $[H]$ are generally themselves not symmetric. Let us consider a simple example which is the problem exterior to a square having a side length equal to 2, as shown in Fig. 1.

Table 1 displays the eigenvalues of matrix $[K]$, showing that one of these eigenvalues is negative, which is physically inconsistent, because it implies a negative supply of energy for some boundary conditions. Now, let us show that this situation is closely related to the well-known problem of loss of uniqueness of the Dirichlet problem which is mentioned in many papers. Indeed, Fig. 2

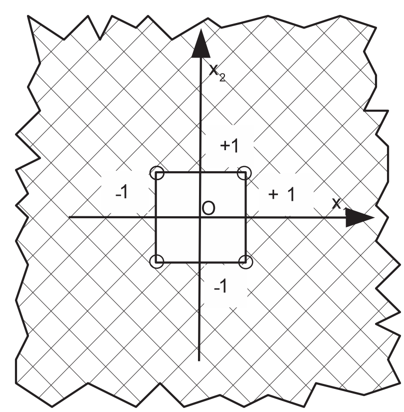

Fig. 1. The exterior 2D problem outside a square.

Table 1
Eigenvalues of the matrix $[K]$ when using the classical 2D Green's function for Laplace equation in the domain of Fig. 1.

Eigenvalues of $[K]$			
−6.06	1.58	1.58	1.72

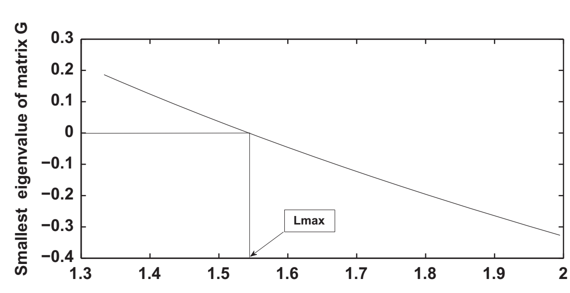

Fig. 2. Smallest eigenvalue of matrix $[G]$ for different side-lengths.