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The finite element implementation of 3D fractional viscoelastic constitutive models

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ABSTRACT

The aim of this paper is to present the implementation of 3D fractional viscoelastic constitutive theory presented in Alotta et al., 2016 [1]. Fractional viscoelastic models exactly reproduce the time dependent behaviour of real viscoelastic materials which exhibit a long "fading memory". From an implementation point of view, this feature implies storing the stress/strain history throughout the simulations which may require a large amount of memory. We propose here a number of strategies to effectively limit the memory required. The form of the constitutive equations are summarized and the finite element implementation in a Newton-Raphson integration scheme is described in detail. The expressions that are needed to be coded in user-defined material subroutines for quasi static and dynamic implicit and explicit analysis (UMAT and VUMAT) in the commercial finite element software ABAQUS are readily provided. In order to demonstrate the accuracy of the numerical implementation we report a number of benchmark problems validated against analytical results. We have also analysed the behaviour of a viscoelastic plate with a hole in order to show the efficiency of these types of models. The source codes for the UMAT and VUMAT are provided as online supplements to this paper.

1. Introduction

In the last decade the use of fractional viscoelastic models has gained interest among researchers as they are capable of accurately represent both creep and relaxation behaviour of viscoelastic materials and the effects of "fading" memory captured experimentally. It has been widely shown that, during a creep/relaxation test, the stress/strain response of viscoleastic materials is characterized by a power law with respect to time; examples are polymers, biological tissues, asphalt mixtures, soils ([2–6]) among others. A power-law in the creep and relaxation responses leads to fractional viscoelastic constitutive models which are characterized by the presence of derivatives and integrals of non-integer order (see Refs. [7,8]). The most attractive aspect of using fractional operators in the viscoelastic constitutive laws is that the stress/displacement response depends on the previous stress/strain history, which allows the long "fading" memory of the material to be taken into account. Another advantage of fractional viscoelastic models is that they are defined by a small number of parameters compared to classical integer order viscoelastic models. Numerous studies have been devoted to theoretical aspects of 1D fractional constitutive laws ([3,9–14]) as well as experimental aspects and parameter characterization ([15–20]) of the constitutive behavior and also application to beam models subjected to both deterministic ([21,22]) and stochastic ([23–25]) conditions. The influence of temperature on the response of fractional viscoelastic models has also been investigated ([26,27]). Some numerical implementation of 1D fractional constitutive laws in finite element codes has been presented (see for example [28]).

3D formulations of fractional viscoelastic models have been proposed and studied (see for example [1,29–32]). In order to be able to use these models to represent the behaviour of real-life engineering components with complex shapes, it is necessary to perform the implementation of these constitutive models into finite element software. To the author's knowledge the implementation of 3D formulations of fractional viscoelastic models in a finite element context is lacking. Indeed, to the best of the authors' knowledge only in Ref. [33] an effort was made to implement fractional viscoelasticity in a finite element code. However, only the fractional standard linear solid (FSLS) model was considered in the paper [33], while many researchers of

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the field use also other fractional viscoelastic model such as the springpot, the fractional Kelvin-Voigt (FKV) model and the fractional Maxwell (FM) model. Hence, the aim of this paper is the implementation of the most used three dimensional fractional viscoelastic constitutive laws in finite element (FE) codes. In particular, we fully cover the details of the implementation of these models in user-defined material subroutines in the commercial finite element software ABAQUS. In our opinion, the numerical implementation of a new fractional viscoelastic theory using the finite element method is often a laborious task especially for researchers new to this area. Here we clearly show the expression of the constitutive tangent tensor that needs to be implemented in the UMAT routine; the implementation is straightforward also for researchers and engineers that have not specific knowledge of fractional calculus. The details of numerical procedures and related expressions that need to be implemented in user defined routines are not extensively published in the literature. Recently, there has been an interest in the dissemination of new computational procedures through publishing research papers addressing all of the aspects related to their implementation. For instance Chester et al. [34] recently presented the implementation of a coupled diffusion mechanics model for elastomeric gels as a user-defined element (UEL) subroutine in ABAQUS. Furthermore, the implementation of a coupled mechanics-diffusion theory in a user defined material routine (UMATHT) in ABAQUS has been presented by Barrera et al. [35] in order to study hydrogen embrittlement mechanisms of steels. Also a cohesive finite element as a UEL subroutine in ABAQUS has also been published by Park and Paulino [36]. Here we show that these 3D fractional viscoelastic models can be easily implemented numerically in a finite element context by using the discretized version of fractional derivatives provided by Grünwald-Letnikov [7]. In this paper we also include the details of computational tools used to access the strain (and/or the stress) history and the possible strategies to reduce the amount of memory required to run analysis of large FE models. This issue has not been discussed elsewhere. The source codes for implicit and explicit analysis of the 3D fractional Kelvin Voigt model are reported as an online supplement to this paper.

The paper is organized as follows: firstly the three-dimensional springpot model is summarized (this is also discussed extensively in Ref. [1]) and then its implementation is described; second, the other fractional viscoelastic models are introduced and their implementation is presented. We then discuss possible solutions to limit the memory required to run large simulations. Finally, comparisons with some benchmark problems are presented in order to show the accuracy of the routines and the possibility to reproduce a wide range of different behaviours.

2. 3D fractional constitutive law

It is well known that a viscoelastic material can be characterized, for one dimensional problems, by its Relaxation and Creep functions $R(t)$ and $C(t)$ respectively. These functions describe the behaviour of the material when a constant strain and a constant stress are applied, respectively.

Experimental tests on real viscoelastic materials, such as polymers, asphalt mixtures, biological tissues, have shown that creep and relaxation are well fitted by power laws of real order rather than exponential functions. In the simplest case in which only one component of the stress is present (hydrostatic or tangential stress), and the creep/relaxation behaviour is well fitted by pure power laws, the relaxation function $R(t)$ and the creep function $C(t)$ are given as [1]:

$$R(t) = \frac{C_\rho t^{-\rho}}{\Gamma(1-\rho)}; \qquad C(t) = \frac{t^\rho}{C_\rho \Gamma(1+\rho)} \tag{1}$$

where $\Gamma(\cdot)$ is the Euler gamma function, ρ is a real number $0 \le \rho \le 1$ and C_ρ is a material parameter evaluated by fitting creep or relaxation experimental curves.

In the frame of linear viscoelasticity, the Boltzmann superposition principle allows us to obtain the response of a material when the imposed stress $s(t)$ or strain history $e(t)$ is not constant and can be expressed in two forms:

$$s(t) = \int_0^t R(t-\tau)\dot{e}(\tau)d\tau \tag{2a}$$

$$e(t) = \int_0^t C(t-\tau)\dot{s}(\tau)d\tau \tag{2b}$$

These integrals are often labelled as "hereditary" integrals, because the actual value of $s(t)$ (or $e(t)$) depends on the entire previous history of $e(t)$ (or $s(t)$). Eqs. (2a) and (2b) are valid for unstrained/unstressed state for $t \le 0$. If $e(0) = e_0 \neq 0$ the term $R(t)e_0$ has to be added in Eq. (2a) or if $s(0) = s_0 \neq 0$ the term $C(t)s_0$ has to be added in Eq. (2b). In the following, without any loss of generality, we suppose that $e_0 = 0$ and $s_0 = 0$.

Substitution of Eq. (1) in Eqs. (2a) and (2b) leads to constitutive laws that involve fractional operators, namely derivatives and integrals of real order ([7], [8]). This is straightforward for the case in which a strain history is applied (Eq. (2a)) and we want to evaluate the corresponding stress history:

$$s(t) = \frac{C_\rho}{\Gamma(1-\rho)} \int_0^t (t-\tau)^{-\rho}\dot{e}(\tau)d\tau = C_\rho \left({}_0^C D_t^\rho e\right)(t) \tag{3}$$

In Eq. (3) the symbol $\left({}_0^C D_t^\rho \cdot\right)$ represents the Caputo fractional derivative ([7]) of order ρ, that is a convolution integral with a power law kernel. In the following sections we will refer to it as $(D^\rho \cdot)$. If we consider the case in which a stress history is applied (Eq. (2b)), integrating by parts and after some manipulations we obtain the Riemann-Liouville (RL) fractional integral of order ρ $\left({}_0D_t^{-\rho}\cdot\right)$ ([7]):

$$e(t) = \frac{1}{C_\rho \Gamma(1+\rho)} \int_0^t (t-\tau)^\rho \dot{s}(\tau)d\tau = \frac{1}{C_\rho \Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} s(\tau)d\tau$$
$$= \frac{1}{C_\rho}\left({}_0D_t^{-\rho} s\right)(t) \tag{4}$$

In the following we will refer to the RL fractional integral as $D^{-\rho}$. The constitutive laws in Eq. (3) and Eq. (4) represent the response of a "springpot" element ([37]). It has been shown in Ref. [9] that the behaviour of the springpot can be reproduced in a classical viscoelasticity framework by an infinite sequence of massless laminae linked by springs/dashpots and laying in a bed of dashpots/springs. This is the reason why the use of fractional viscoelasticity results in a significant reduction of mechanical parameters compared to using calssical viscoelastic models.In order to model the isotropic three-dimensional behaviour of the springpot, it is sufficient to define two relaxation (or creep) functions. The most convenient choice is to use volumetric and deviatoric relaxation (or creep) functions. The relaxation matrix can be written as follows:

$$R_{ijkh}(t) = \left(K_R(t) - \frac{2}{3}G_R(t)\right)\delta_{ij}\delta_{kh} + G_R(t)\left(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\right) \tag{5}$$

where δ_{ij} is the Kronecker symbol. For both deviatoric $G_R(t)$ and volumetric relaxation functions $K_R(t)$, power law functions analogous to first of Eq. (1) are selected:

$$G_R(t) = \frac{G_\alpha t^{-\alpha}}{\Gamma(1-\alpha)} \tag{6a}$$

$$K_R(t) = \frac{K_\beta t^{-\beta}}{\Gamma(1-\beta)} \tag{6b}$$

where K_α and G_β are anomalous bulk and shear relaxation moduli, respectively, while α and β are real numbers indicating the orders of bulk and shear power laws, respectively.