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Spectral investigations of Nitsche's method

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ABSTRACT

Incompatible discretization methods provide added flexibility in computation by allowing meshes to be unaligned with geometric features and easily accommodating non-interpolatory approximations. Such formulations that are based on Nitsche's approach to enforce surface constraints weakly, which shares features with stabilized methods, combine conceptual simplicity and computational efficiency with robust performance. The basic workings of the method are well understood, in terms of a bound on the parameter. However, its spectral behavior has not been explored in depth. Such investigations can shed light on properties of the operator that effect the solution of boundary-value problems. Furthermore, incompatible discretizations are rarely used for eigenvalue problems. The spectral investigations lead to practical procedures for solving eigenvalue problems that are formulated by Nitsche's approach, with bearing on explicit dynamics.

1. Introduction

Incompatible discretization methods such as [1–4] accommodate non-conforming meshes, allowing elements to be unaligned with geometric features such as domain boundaries or internal interfaces, and incorporating non-interpolatory approximations which can account for features of the solution in the analysis. Their primary goal is to increase the geometric flexibility of discretization schemes and to alleviate meshing related obstacles in complex and evolving configurations, tasks considered to be among the most difficult, labor intensive, and time consuming in finite element computation. Using these methods requires special attention to practical issues such as conditioning [5].

At the heart of incompatible discretization lies the treatment of surface constraints. Nitsche's method for enforcing surface constraints weakly, based on stabilized variational formulations, leads to efficient procedures for embedding kinematic boundary and interface conditions in computational meshes. Nitsche's method was originally developed as a variationally consistent penalty method for weakly enforcing Dirichlet boundary conditions in second-order problems [6], but is perhaps more constructively interpreted [7] as related to stabilized hybrid methods [8]. Nitsche's formulation is conceptually simple and free of auxiliary fields, thereby reducing computational cost. Variational consistency provides robust performance with respect to the value of the method parameters, along with rational procedures to determine the parameters. Similar approaches were proposed for exterior acoustics problems [9,10].

This work presents an initiatory exploration of the spectral behavior of Nitsche's method, and proposes procedures for solving eigenvalue problems formed by this approach. On a given (conforming) mesh, weak imposition of kinematic boundary conditions gives rise to additional degrees of freedom compared to the standard approach of enforcing them as admissibility requirements, and hence, additional eigenpairs. In contrast to the eigenpairs of the standard discrete formulation, which approximate a finite number of exact eigenpairs, the additional ones are associated with enforcing the boundary constraints. The constraint eigenvalues are indefinite in the absence of stabilization. In practice, the Nitsche stabilization parameter should be defined to ensure coercivity of the discrete operator [11,13]. It follows that the constraint eigenvalues depend on the stabilization, raising the question whether the physical ones do as well. Subsequent investigations show that the constraint eigenvalues grow linearly with the stabilization, while the physical ones are virtually constant.

As increasing constraint eigenvalues approach physical eigenvalues, they may cross or veer. Eigenvalue veering is known (under different names) in diverse disciplines [14,15]. It occurs typically in systems with parameters, continuous as well as discrete. The phenomenon is characterized by rapid and opposite change in curvatures. Eigenvalue veering reflects coupling of eigenfunctions, which are interchanged. In the

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veering zones, eigenfunctions of veering eigenvalues form linear combinations, and the eigenpairs lack clear physical or constraint character. This behavior is conspicuous, yet ultimately it does not impair the performance of the method.

Incompatible formulations are rarely used for eigenvalue problems, possibly due to the potential presence of constraint eigenpairs. (For an example of spectral analysis using an incompatible method see Ref. [16].) Removing the added degrees of freedom on the boundaries addressed by the Nitsche formulation by algebraic elimination yields a system that is free of the constraint eigenpairs, and hence may be solved by any conventional eigenvalue solver. Using Irons-Guyan reduction [17,18] for this purpose is relatively inexpensive and preserves the structure of the original formulation, to a large extent.

Practical issues related to non-conforming meshes such as potential ill conditioning in the absence of special measures [5,19,20] are not addressed in this initiatory investigation.

In Section 2 we present the variational formulation for the eigenvalue problem for the Laplacian using Nitsche's method. Guided by the Rayleigh quotient, we define a boundary quotient, show that it equals the derivative of the eigenvalue with respect to the stabilization parameter, and use it to identify the constraint eigenpairs.

In Section 3 we investigate numerically the spectrum arising from Nitsche's formulation on conforming meshes in two sample problems. We show that constraint eigenfunctions are mesh-dependent quantities, restricted to the neighborhood of the Nitsche boundary. The spectrum of a reduced system obtained by algebraic elimination contains only physical eigenpairs, and is free of veering.

In Section 4 we propose practical procedures for solving eigenvalue problems formulated by Nitsche's method.

2. An eigenvalue problem for the Laplacian

Let Ω be an open, bounded region with boundary Γ. The Dirichlet eigenvalue problem for the Laplacian Δ is

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{2.1}$$

$$u = 0 \quad \text{on } \Gamma \tag{2.2}$$

The Dirichlet problem is presented for simplicity. Consideration of other types of boundary conditions is straightforward.

The nontrivial solutions of this problem are a countably infinite number of eigenpairs $\{\lambda, u\}$. The real, positive eigenvalues λ are considered to be ordered in ascending value.

2.1. The Nitsche formulation

Nitsche's approach to enforce surface constraints weakly is useful for boundary and interface conditions. We consider the eigenvalue problem for the Laplacian, with Dirichlet boundary conditions enforced by Nitsche's method, stated in terms of functions that are free of kinematic admissibility requirements

$$a(w,u) - \lambda(w,u) = 0 \tag{2.3}$$

Here,

$$a(w,u) = (\nabla w, \nabla u) - (w, u_{,n})_\Gamma - (w_{,n}, u)_\Gamma + \alpha (w,u)_\Gamma \tag{2.4}$$

$(\cdot, \cdot)$ is the $L_2(\Omega)$ inner product with subscripts denoting other domains of integration, ∇ is the gradient, a comma is used for differentiation with $(\cdot)_{,n}$ denoting the normal derivative, and α is the stabilization parameter. If kinematically admissible functions are used, this reduces to the underlying standard formulation.

In the discrete settings, setting the stabilization parameter on the element level to a value larger than the constant of a suitable discrete trace inequality C recovers the coercivity of the standard formulation [7] and provides good performance in practice [11–13]. This determines the recommended range of operation of the Nitsche method.

The standard discrete formulation yields a finite number of eigenpairs referred to as 'physical' since they approximate the lower eigenpairs of the continuous problem. The approximate eigenvalues overestimate the exact (positive) values. On the same (conforming) mesh, Nitsche's formulation has more degrees-of-freedom than the standard formulation (on the boundary), and therefore additional eigenpairs. These are associated with the constraints, and are not approximations of exact eigenpairs. The constraint eigenvalues are indefinite for $\alpha < C$. The approximate physical eigenvalues need not overestimate the exact values.

A central goal of this work is to separate the two types of eigenpairs that arise in the Nitsche formulation. The sign of the eigenvalues is inadequate for this purpose since some of the constraint eigenvalues are positive for all values of α, and all are positive for $\alpha > C$.

2.2. Parameter sensitivity

For a specific eigenpair $\{\lambda, u\}$

$$a(u,u) - \lambda(u,u) = 0 \tag{2.5}$$

Recall the Rayleigh quotient

$$\mathcal{R}(v) = \frac{a(v,v)}{\|v\|^2} \tag{2.6}$$

Here, $\|v\|$ is the $L_2(\Omega)$ norm, with subscripts subsequently denoting other domains of integration. The eigenvalue is equal to the Rayleigh quotient of the corresponding eigenfunction $\lambda = \mathcal{R}(u)$. Along similar lines, define a *boundary quotient*

$$\mathcal{B}(v) = \frac{\|v\|_\Gamma^2}{\|v\|^2} \tag{2.7}$$

Differentiating (2.5) with respect to α

$$a(u_{,\alpha}, u) - \lambda(u_{,\alpha}, u) + \|u\|_\Gamma^2 - \frac{d\lambda}{d\alpha}\|u\|^2 + a(u, u_{,\alpha}) - \lambda(u, u_{,\alpha}) = 0 \tag{2.8}$$

The first two terms vanish by (2.3) (since $u_{,\alpha}$ is an admissible test function) and the last two terms vanish by symmetry and (2.3). Thus the sensitivity of an eigenvalue to the Nitsche parameter is equal to the boundary quotient of the corresponding eigenfunction

$$\frac{d\lambda}{d\alpha} = \mathcal{B}(u) \tag{2.9}$$

It follows that

$$\frac{d\lambda}{d\alpha} \geq 0 \tag{2.10}$$

This result is in agreement with the intuitive notion of stabilization as effective stiffness (related to the Laplacian in this case), in that increasing the stabilization gives rise to higher eigenvalues. However, this simplistic perception is impaired by the intricate eigenvalue behavior observed in the subsequent numerical studies, identifying two types of eigenpairs which are interchanged at times.

3. Numerical studies

The spectral features of Nitsche's method are explored by numerical studies in two sample problems. All of the eigenpairs are computed in these studies of spectral behavior. For direct comparison to the standard formulation, conforming meshes are considered, namely elements are aligned with geometric features and approximations are interpolatory. Subsequent numerical tests are performed on structured meshes of bilinear Lagrange elements ($C = 1/h$), unless noted otherwise.