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Virtual element formulation for isotropic damage

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A R T I C L E I N F O

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A B S T R A C T

In the paper we present a low-order virtual element formulation for modelling the strain-softening response of quasi-brittle materials. For this purpose, a formulation in two-dimensions is considered, with virtual elements having arbitrary shape. The method is based on minimization of an incremental energy expression, with a novel construction of the stabilization energy for isotropic elasto-damage. A set of numerical examples, illustrating the efficiency of the proposed method, complements the paper.

1. Introduction

The Virtual Element Method (VEM) has been recently developed in keeping with Mimetic Finite Difference [12], characterizing it as a Galerkin finite element-type re-formulation. The basic principles have been presented in the seminal work by Ref. [5]. Subsequent significant contributions in explaining the theoretical basis and providing examples of implementations can be found in Refs. [7,13,19].

The VEM permits the numerical solution of boundary value problems on arbitrary polyhedral meshes, including convex and non-convex elements, very stretched elements, hanging nodes and collapsing nodes. The great flexibility in dealing with very general geometries and the robust mathematical basis of the method pave the way for possible use in very general cases. They are ranging from crack propagation in fractured solids, to modelling the texture evolution in polycrystalline materials, up to reproducing the complex behaviour of structured materials. Applications have been devoted so far to linear elastic two- and three-dimensional problems [1,6,19], discrete fracture network simulations [10], eigenvalue problems [29], contact problems [45], topology optimization [20] and nonlinear problems [2,8,14]. Stabilization procedures for the virtual element method, which are well known from the work of [9] for finite elements, are described in Ref. [13] for linear Poisson problems. In the VEM formulation a stabilization term is mandatory. The structure of the VEM, indeed, typically comprises a term in the weak formulation or energy functional in which the quantity ϕ_v, here deformation, is replaced by its projection $\Pi\phi_v$ onto a polynomial space. This results in a rank-deficient structure, so that it is necessary to add a stabilization term to the formulation, see Refs. [5,6,14], where in the latter the scalar stabilization parameter of the linear case being replaced by one that depends on the fourth-order elasticity tensor.

In the framework of compressible and incompressible nonlinear elasticity, a novel stabilization technique has been proposed in Ref. [44], inspired by an idea first proposed in Ref. [30], generalized in Ref. [11] and simplified in Ref. [24]. The key innovative aspect is to add to the positive semidefinite mean strain energy, Ψ, a positive-definite energy, $\widehat{\Psi}$, evaluated using full quadrature. Moreover, for consistency a term involving $\widehat{\Psi}$ as a function of the mean strain is subtracted. The resulting strain energy is the sum of the original energy as a function of the projected displacement, $\Psi(\Pi\boldsymbol{u}_v)$, and the term $\widehat{\Psi}(\Pi\boldsymbol{u}_v)$ to which a positive definite stabilization energy as a function of the displacement and its projection are respectively added and subtracted. In the terms involving $\widehat{\Psi}(\boldsymbol{u}_v)$, the quadrature is carried out by constructing a triangular mesh in the element, without introducing additional degrees of freedom, since the nodal points are those of the original element. The same stabilization has been successfully applied also to problems of finite strain plasticity [43].

In the present paper, this stabilization technique is modified and exploited in the framework of a virtual element formulation for 2D scalar damage problems.

The main idea is to test the effectiveness of the VEM in dealing with highly localized strains due to material instabilities, typically exhibited by quasi-brittle materials undergoing severe loading conditions.

In the last decades, many efforts have been focused on the formulation of continuum damage models able to predict the irreversible phenomena related to the onset and evolution of void formation,

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micro-cracking and strain-softening in quasi-brittle materials. Based on Kachanov's pioneered work [22], different phenomenological models have been proposed both accounting for isotropic [16,25,27,33,40,41], or anisotropic [21,26,34,41] damage response. Special attention has been devoted to develop strategies able to overcome the well-known spurious mesh sensitivity problems, occurring in finite elements computations when, in the presence of softening behaviour, the governing differential equations may loose ellipticity and resort to ill-posed boundary value problems. Successful remedies have been proposed by recourse to non-local continuum theories [17,18], which possess intrinsic regularization properties due to the natural introduction of characteristic lengths, or either recourse to viscous regularization [31,42]. Alternative solutions to the pathological mesh dependency in damage models are non-local constitutive models, see e.g. Ref. [3] for a comprehensive survey, or local models properly enriched by a dependence of the material properties on the element size, as in Refs. [32,33,40].

We focus here on an isotropic damage law, in the framework of linearised kinematics, with different thresholds in tensions and compression, as proposed in Ref. [33], and we adopt as regularization technique, alternatively, the last two aforementioned approaches to which we will refer as "nonlocal" and "local".

The paper is organized as follows: in Section 2 the governing equations of the local and non-local scalar damage model are briefly recalled; Section 3 is devoted to the construction of the linear ansatz functions of the proposed virtual element approach; the virtual element formulation is, then, fully developed in Section 4; a number of illustrative applications is proposed in Section 5, and, finally, in Section 6 some concluding remarks are reported.

2. Governing equations for isotropic damage model

Consider an initially elastic body that occupies the bounded domain $\Omega \subset \mathbb{R}^2$. Let $\Gamma = \Gamma_D \cup \Gamma_N$ be the boundary of Ω, with Γ_D the Dirichlet and Γ_N the Neumann boundaries, such that $\Gamma_D \cap \Gamma_N = \emptyset$. Each material point $\mathbf{x}$ is characterized by the displacement field $\mathbf{u}$ that is the primary unknown variable. The symmetric Cauchy stress $\boldsymbol{\sigma}$ satisfies the linear momentum balance

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f}, \tag{1}$$

with $\mathbf{f}$ being the body force.

The Dirichlet and Neumann boundary conditions hold, respectively, as:

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}}, \qquad \text{on} \quad \Gamma_D \\ \boldsymbol{\sigma}\mathbf{n} &= \mathbf{t}, \qquad \text{on} \quad \Gamma_N. \end{aligned} \tag{2}$$

with $\bar{\mathbf{u}}$ the prescribed displacement, $\mathbf{n}$ the outward unit normal vector and $\mathbf{t}$ the surface traction.

The strain-displacement relation is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}). \tag{3}$$

For isotropic damage a scalar variable d is introduced, satisfying $0 \le d \le 1$. At a given point of the damaged material, the free energy is expressed as a function approaching zero as the damage d increases:

$$\Psi(\boldsymbol{\varepsilon}, d) = (1-d)\Psi^0(\boldsymbol{\varepsilon}). \tag{4}$$

Here Ψ^0 is the initial undamaged elastic energy that reads as

$$\Psi^0(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^T\mathbb{D}\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}_{EL}. \tag{5}$$

where $\boldsymbol{\sigma}_{EL} = \mathbb{D}\boldsymbol{\varepsilon} = \lambda tr(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$ is the stress for an initially homogeneous isotropic elastic material, with $\mathbb{D}$ being the elasticity tensor, and λ and μ the Lamé constants.

The energetic consistency of the constitutive model is ensured by the fulfilment of the Clausius-Planck inequality [28],

$$\dot{D} = \left(\boldsymbol{\sigma} - \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\right)^T\dot{\boldsymbol{\varepsilon}} - \frac{\partial\Psi}{\partial d}\dot{d} \ge 0 \tag{6}$$

where $\dot{D}$ is the rate of the mechanical energy dissipation defined for arbitrary infinitesimal variations $\dot{\boldsymbol{\varepsilon}}$. Using the Coleman's method [15] it follows that the constitutive relation is

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = (1-d)\mathbb{D}\boldsymbol{\varepsilon} \tag{7}$$

and

$$\dot{D} = -\frac{\partial\Psi}{\partial d}\dot{d} = -\Psi^0\dot{d} \ge 0. \tag{8}$$

Following [33,41], an equivalent effective stress τ is defined as a suitable energy norm of the undamaged stress tensor $\boldsymbol{\sigma}^0$ and used to compare different material states. Here we adopt a damage model with different thresholds in tension and compression. Hence we choose τ as

$$\tau = \left(\zeta + \frac{1-\zeta}{n}\right)\sqrt{\Psi^0(\boldsymbol{\varepsilon})}, \quad \text{with} \quad \zeta = \frac{\sum_{i=1}^3 \langle\sigma_i^0\rangle}{\sum_{i=1}^3 |\sigma_i^0|}, \tag{9}$$

where ζ is a weight factor depending on the elastic principal stresses σ_i^0, $\langle\bullet\rangle$ is the Macaulay bracket, and $n = f_c/f_t$ is the ratio between compressive f_c and tensile f_t strength of the material.

The damage criterion is defined via the limit damage surface, i.e. a function $F(\tau^t, r^t)$ that splits the admissible stress space into the elastic domain (when $F < 0$) and the damage domain (when $F = 0$). It depends both on the equivalent effective stress τ^t and on a material parameter representing the damage threshold r^t at the current time t.

The most general form of $F(\tau, r)$ among different possibilities is

$$F(\tau^t, r^t) = G(\tau^t) - G(r^t), \quad \forall t \ge 0, \tag{10}$$

where $G(\bullet)$ is a suitable monotonic scalar function.

The damage is governed by the following evolution equations

$$\dot{r} = \dot{\gamma}, \quad \dot{d} = \dot{\gamma}\frac{\partial F}{\partial\tau}, \tag{11}$$

with $\dot{\gamma}$ being the damage multiplier adopted to define the loading-unloading conditions (equivalent to the plastic multiplier in the rate independent plasticity). The following Kuhn-Tucker relations have also to be satisfied

$$F(\tau, r) \le 0, \quad \dot{\gamma} \ge 0, \quad \dot{\gamma}F(\tau, r) = 0. \tag{12}$$

It is possible to directly integrate the evolution of the internal variables, as in Refs. [33,41], and obtain

$$r^t = \max\left(r^0, \max(\tau^s)\right), \quad 0 \le s \le t \quad d = G(r^t). \tag{13}$$

where r^0 is a characteristic of the material, i.e. the initial damage threshold for the virgin material and $\max(\tau^s)$ is the maximum equivalent effective stress attained until current time t.

Among different possible choices of explicit functions for the scalar damage d, two options are frequently used in literature, e.g. Refs. [33,40]:

1. Linear damage law

$$d_1(r^t) = \frac{1}{1+H_1}\left(1 - \frac{f_t}{r^t}\right), \quad f_t \le r^t \le \infty, \tag{14}$$

where $H_1 = f_c^2/(2n^2G_fE^0)$, E^0 is the elastic modulus, and G_f the fracture energy per unit area.

2. Exponential damage law

$$d_2(r^t) = 1 - \frac{f_t}{r^t}\exp\left[H_2\left(1 - \frac{r^t}{f_t}\right)\right], \quad f_t \le r^t \le \infty, \tag{15}$$

where $H_2 = \left(\frac{n^2G_fE^0}{f_c^2} - \frac{1}{2}\right)^{-1} \ge 0.$