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Exploring the design space of nonlinear shallow arches with generalised path-following

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ABSTRACT

The classic snap-through problem of shallow arches is revisited using the so-called *generalised path-following technique*. Classical buckling theory is a popular tool for designing structures prone to instabilities, albeit with limited applicability as it assumes a linear pre-buckling state. While incremental-iterative nonlinear finite element methods are more accurate, they are considered to be complex and costly for parametric studies. In this regard, a powerful approach for exploring the entire design space of nonlinear structures is the generalised path-following technique. Within this framework, a nonlinear finite element model is coupled with a numerical continuation solver to provide an accurate and robust way of evaluating multi-parametric structural problems. The capabilities of this technique are exemplified here by studying the effects of four different parameters on the structural behaviour of shallow arches, namely, mid span transverse loading, arch rise height, distribution of cross-sectional area along the span, and total volume of the arch. In particular, the distribution of area has a pronounced effect on the nonlinear load-displacement response and can therefore be used effectively for elastic tailoring. Most importantly, we illustrate the risks entailed in optimising the shape of arches using linear assumptions, which arise because the design drivers influencing linear and nonlinear designs are in fact topologically opposed.

1. Introduction

Structural nonlinearities, particularly those of an elastic nature, are gaining considerable momentum within engineering applications, and are being viewed as a positive design feature [1]. Nonlinear structural problems have been discussed in the literature for decades, however they are seldom exploited outside of the academic environment. This general reluctance engineers harbour for nonlinear structures is justified by two prevailing statements: (i) the lack of sufficiently robust computational tools, and (ii) the time-consuming nature of solving incremental-iterative problems, especially when multi-parametric studies for optimisation or imperfection sensitivity are conducted.

Across all length scales, slender and thin-walled structures are commonly used in engineering applications for a number of reasons. In micro- and meso-scale applications thin-walled structures are exploited for their ease of manufacture and ability to deform significantly without failure, thus providing unparalleled functionality that relies on nonlinear behaviour. In macro-scale applications, such as the aerospace and automotive sectors, thin-walled shell structures are used for their structural efficiency.

Although more efficient in their load carrying capacities, macro-scale thin-walled and slender structures are susceptible to structural instabilities. Small-scale structures relying on nonlinearities for functionality and large scale structures being prone to instabilities, it is evident that geometric nonlinearities need to be accounted for at all stages of the design process.

Arched structures, which are the focus of this paper, are known to exhibit instabilities and are a textbook example of classic snap-through buckling behaviour. In fact, snap-through instabilities of arched structures occur in a wide range of applications, from the tailorable design of micro-electromechanical systems (MEMS), to the meso-scale light-switch type structures, and further, to the failure of macro-scale civil structures.

Large scale arched structures, particularly those whose loading capacity is intrinsically linked to the deformations sustained, are particularly prone to instabilities. Carpinteri *et al.* [2] present an industrial

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example illustrating the importance of nonlinear analysis for a modern roof-structure. Carpinteri concludes that current linear methods are insufficient in predicting the post-buckled state and, most importantly, the load-carrying capacity of the roof.

The same buckling behaviour is also observed on the micro-scale within the field of MEMS [3–5], a discipline which has seen significant growth in recent years. Micro-electromechanical systems provide a particularly challenging problem since the topic couples a number of disciplines, from solid and fluid mechanics to thermomechanics and electromagnetism. All of these fields can introduce their own forms of nonlinearity and these can interact in a complex manner to yield emergent phenomena that are difficult to predict.

Established methods for exploring a nonlinear design space require onerous parametric studies. With the application of the generalised path-following technique, however, the design space can be evaluated within a single solution process, for any number of parameters.

1.1. History of arched beam structures

Concave load-bearing structures are one of the oldest structures known to man. In this sense, a clear demarcation between *masonry arches* introduced in antiquity, *i.e.* concave structures constructed by a series of rigid building blocks joined with little to no tensile loading, and *elastic arches*, capable of resisting both membrane forces and bending moments, is necessary. For a fascinating history of arch construction, and its theoretical development from arch theory to computational mechanics, the interested reader is directed to chapter 4 of reference [6]. Throughout this paper we refer to compliant elastic arches which use snap-through for functionality and hence the research presented herein is restricted to slender arches. Such compliant arches, which utilise elastic snap-through "failure" well before plastic deformations occur, are being used in MEMS devices [7] and for novel metamaterials [8].

The critical buckling of shallow arches, either by symmetric snap-through or by an asymmetric bifurcation, is a seemingly well-understood problem. Once the solution space is opened up to more parameters beyond a simple load factor, however, it quickly becomes apparent that this problem is more intricate and complex than at first sight.

1.2. Generalised path-following in structural mechanics

There appears to be little question that the so-called incremental-iterative methods represent by far the most popular procedures for the solution of nonlinear continuum mechanics. Conventional path-following techniques are based on a single parameter, either: displacement [9,10], load [11,12], external work [13,14], arc-length [15–18], or others [19], and these result in a single load-displacement curve, as illustrated in Fig. 1a. This curve is, however, only a single equilibrium locus on a multi-dimensional solution manifold parametrised by any number of other variables that can influence the behaviour of the structure, *e.g.* material properties, geometric dimensions, imperfections, *etc.* Hence, traditional arc-length methods available in commercial finite element solvers are degenerate cases of a *generalised path-following technique* restricted to the forcing parameter-displacement space, and it is cumbersome or impossible to change two or more parameters simultaneously, which for example is required for tracking bifurcations. Whereby the only parameter that can be actively varied in these commercial solvers is the loading factor. The capabilities of a generalised path-following technique exceed those of conventional path-following methods by enabling visualisation of the structure's behaviour in multi-dimensional space. This technique allows *any* number of parameters to be continued, *i.e.* treated akin to a loading factor, during a single solution run and thus eliminates the necessity for extensive parametric studies.

Historically, the generalised path-following technique has been used extensively in the fields of applied mathematics and physics [20–23], where the term *numerical continuation* is a more common designation. In structural engineering applications, however, *path-following* is a familiar term and therefore *generalised path-following*, as introduced by Eriksson and co-workers [24], is a more intuitive designation as it differentiates from conventional path-following in load-displacement space.

In the 1960's Sewell introduced the notion of the equilibrium surface [25], whose shape could be used to identify the stability of the underlying structure with respect to changes in the governing parameters. With the advent of catastrophe theory in structural mechanics this interest intensified, mostly in an analytical setting [26–28], but a generalised computational framework was not introduced to the community until the 1980's by Rheinboldt [29–31]. The concepts introduced by

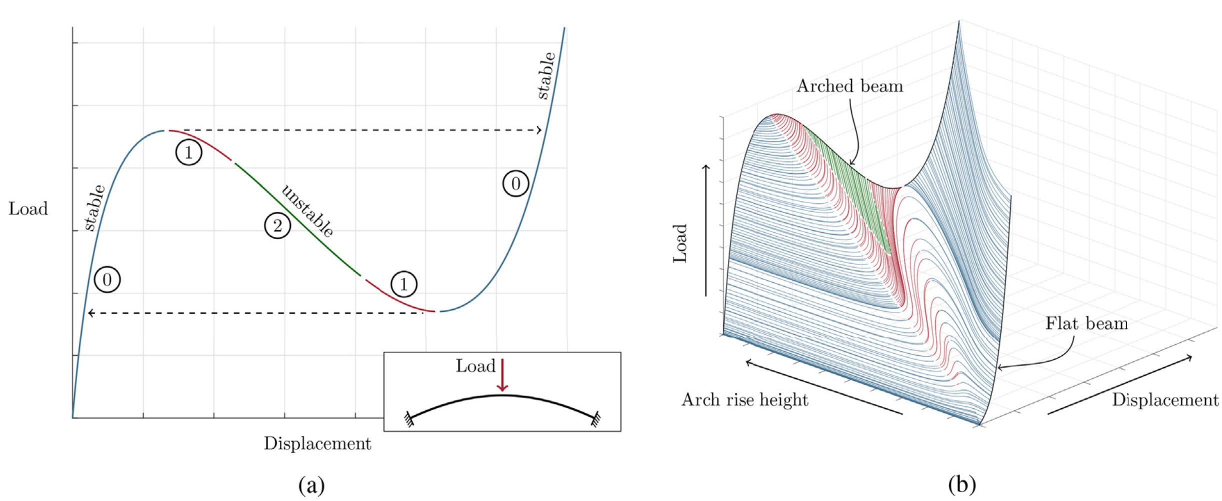

Fig. 1. (a) Fundamental equilibrium path for a baseline case of an arched beam, illustrating classic snap-through behaviour. Numbers and colours on curve segments denote the degree of instability; (b) A solution surface created by the generalised path following technique, here the arch rise height is varied from zero to a predefined maximum. The colours on curve segments, once again, denote the degree of instability. On both figures the different colour segments are separated by critical points, where the stability of the structure with respect to the loading parameter changes. The colour blue denotes a stable equilibrium solution, red denotes an unstable equilibrium solution with one negative eigenvalue of the tangential stiffness matrix, green denotes an unstable equilibrium solution with two negative eigenvalues of the tangential stiffness matrix. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)