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On the locking free isogeometric formulations for 3-D curved Timoshenko beams

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ABSTRACT

Locking-free isogeometric formulations of 3-D curved Timoshenko beams are studied. In particular, the global $\overline{B}$ projection method and a mixed formulation are examined and compared, showing equivalently optimal convergence with regard to displacement solutions. These two formulations are proved to be equal when the beam cross-sectional properties are uniform. For beams with non-uniform cross-sectional properties, the mixed formulation is superior in terms of stress recovery. In addition to these two methods, an alternative locking-free formulation, a C^0 NURBS element with selectively reduced integration is suggested in this study, making use of the traditional selective reduced integration (SRI) rule designed for Lagrangian elements. This SRI C^0 NURBS element is simple to implement, preserves the sparsity of the global stiffness matrix and requires fewer quadrature point evaluations. Most importantly, due to the use of NURBS basis functions, the exact curve geometry is preserved with all three locking-free isogeometric elements. Benchmark problems and illustrative examples with complex curved geometries are examined for a detailed investigation of the considered locking-free elements.

1. Introduction

Modeling of curved beams is an important topic that has been addressed by many studies in the past [1–3]. Geometries of curved beams in a computer aided design (CAD) environment are usually modeled by B-splines or Non-Uniform Rational B-Splines (NURBS). On the other hand, these geometries are approximated by linear or quadratic elements in standard finite element analysis (FEA). Such a representation is approximate, however, as curvature and torsion of complex 3-D beams cannot be exactly represented by linear or quadratic elements. To bridge the gap between CAD and FEA, Hughes et al. [4] proposed isogeometric analysis (IGA) wherein the exact geometrical forms from CAD can be preserved in analysis models. This feature of IGA has been shown to improve the accuracy of the numerical simulations in the previous studies [5–7]. Besides, the increased smoothness of higher order NURBS bases has also been explored in the literature and shown to be useful for higher order theories that requires high continuity [8–10]. In a previous study [11], an isogeometric 3-D curved Timoshenko beam of arbitrary shape has been proposed, showing the promise of using IGA for the analysis of curved structural components with arbitrarily complex geometries. Similar to the classical Lagrangian finite element, however, the NURBS elements are prone to shear and membrane locking when the beam becomes slender and the curvature of the beam centroid curve becomes large. Shear and membrane locking appear due to the inability of the interpolation functions to represent the cases of shearless bending and inextensible bending. As shown in Refs. [11,12], higher-order NURBS bases help to alleviate locking without any difficulties because of the fact that higher order NURBS bases are stable due to their variational diminishing property, which is not the case for Lagrangian bases. Although higher-order NURBS curves can be generated through p-refinement or k-refinement in a robust and efficient way, the use of higher order basis functions inevitably increases the computational cost as the number of quadrature points and the bandwidth of the stiffness matrix are both increased. Furthermore, higher order bases cannot eliminate locking completely, as demonstrated in Refs. [11,13]. Thus, there is a need for the development of locking-free NURBS elements.

Various methodologies exist in the literature for alleviating locking in standard finite elements, including selective reduced integration [14], B-bar projection [15], the discrete strain gap method [16,17], mixed formulations [1,18], and discontinuous Galerkin methods [19], among others. Some of these ideas have been successfully extended to isogeometric analysis, for example in Ref. [20] where a selective reduced integration (SRI) rule was proposed for the elimination of both shear and membrane locking in planar curved Timoshenko beams with up to 2nd-order NURBS basis functions. This SRI rule for IGA, however, becomes rather complex when considering higher order basis functions with non-uniform inter-element continuity. When considering only uniform inter-element continuity within one patch, Adam et al. [13] extended the

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SRI rule to higher order NURBS bases. This assumption can be rather restrictive as curves with varying smoothness are rather common in real engineering practices. The so-called discrete strain gap (DSG) method was also extended to IGA by Echter et al. [12] for eliminating shear locking in straight Timoshenko beam elements. When applied to curved beam elements, however, this approach shows less accuracy as compared to other methods such as SRI and B-bar projection [20]. Moreover, this method also requires the inversion of a fully-populated global stiffness matrix, increasing the computational cost as compared to SRI.

Compared to the aforementioned SRI and DSG methods, the B-bar projection method has been shown to be more general and robust in handling locking for problems with various NURBS formulations [20]. The B-bar projection method was introduced for IGA by Elguedj et al. [21] to address volumetric locking and later applied to planar curved Timoshenko beam elements by Bouclier et al. [20] to alleviate shear and membrane locking. By projecting the membrane and shear strains onto a space spanned by B-spline basis functions of one order lower than the solution space, this method can completely eliminate locking phenomena. However, this method again results in a fully populated stiffness matrix, precluding the use of sparse solvers and thus increasing the computational burden. In an attempt to employ the generality and robustness of the B-bar projection method while also reducing the computational cost of inverting a fully-populated matrix, a local B-bar projection method was proposed [22]. This method carries out projection at the element level instead of at the patch level and was extended to curved Timoshenko beam elements by Miao et al. [23] to address membrane and shear locking problems. While the sparsity of the stiffness matrix is preserved with this method, the uniform convergence exhibited by the global projection method may be lost and locking phenomena may still be exhibited with coarse meshes [23]. It should be mentioned that a similar local $\overline{B}$ projection approach has been used in Refs. [24,25] to eliminate membrane locking in Kirchhoff beams and Kirchhoff-Love shells. An alternative local projection strategy was proposed by Hu et al. [26], and though the results presented therein are promising, specific rules for the selection of the projection spaces for NURBS bases with varying inter-element continuity are not obvious.

As an alternative way of formulating locking-free finite elements, mixed methods have been utilized successfully with standard finite elements [1]. For C^0 Lagrangian elements, the stress (or force) fields are interpolated independently using a space spanned by bases of one order lower. Due to the discontinuity of this space, the stress field can be eliminated at the element level and a global displacement-based formulation is recovered. For IGA, this elimination can only be done at the patch level due to the increased inter-element continuity. This results in a fully-populated stiffness matrix in the same way as for the B-bar projection method. In fact, the two methods were shown to be equivalent for two 2-D case [20]. In addition to Galerkin formulations, collocation methods have also been combined with mixed element formulations to alleviate locking phenomena in IGA. By choosing Greville abscissae as the collocation points, this combination was shown to be efficient in addressing shear locking in straight beam elements [27] and both shear and membrane locking in spatially curved beam elements [28]. Among locking-free formulations utilizing the Galerkin method, the global B-bar projection and mixed element formulations share perhaps the highest accuracy per degree of freedom, and so are preferred despite the computational cost of inverting fully populated matrices. Although the SRI method in Ref. [20] preserves high accuracy with relatively low computational cost (due to the sparsity of the stiffness matrix), it is based on complicated rules to deal with varying inter-element continuity for higher order NURBS bases, and so is less desirable. It is also noted that in the studies by Ishaquddin et al. [29,30], two new locking phenomena – flexure locking and torsion locking – were recognized and addressed. Flexure locking occurs when the flexural stiffness is far greater than the torsional stiffness, i.e. $EI \gg GJ$, and is due to the inability of the numerical solution in capturing flexureless torsion [31]. Torsion locking, on the other hand, is manifested in the case of $GJ \gg EI$, which is due to the inability of solution spaces in representing torsionless flexure. For curved beams with regular cross-section shapes and usual engineering materials, these two cases rarely occur, however. Thus, these two locking cases are not considered in the present study.

This study aims to examine and compare the global B-bar projection method and a mixed formulation for addressing locking in 3-D curved Timoshenko beam elements, with a detailed discussion on the choice of projection space and mixed interpolation space. These two methods are shown to be equal under certain conditions. Moreover, for beams with non-uniform cross-sectional properties, the mixed formulation is shown to be superior for stress recovery. As an alternative way to address locking in 3-D curved beam elements, a C^0 NURBS element with selectively reduced integration is presented. This element—though of lower accuracy per degree of freedom than the B-bar projection and mixed formulations—is simple to implement, preserves the sparsity of the stiffness matrix and requires fewer quadrature point evaluations. The outline of the paper is as follows: In Section 2, the curved Timoshenko beam theory is presented and in Section 3 a brief review of IGA is provided. Three locking-free NURBS element formulations are discussed in Section 4. In Section 5, four numerical examples are used to evaluate and compare the performance of the three locking-free formulations. Important conclusions of this study are presented in Section 6.

2. Curved Timoshenko beam theory

2.1. Frenet-Serret bases

The theoretical framework for 3-D curved beams is based on the differential geometry of curves embedded in $\mathbb{R}^3$. Thus, provided that the centroid line of beam cross-sections is continuously differentiable, it can be represented by a local orthonormal bases called the Frenet-Serret bases. Let $\boldsymbol{r}(s) : [0, L] \rightarrow \mathbb{R}^3$ be an arbitrary continuously differentiable curve embedded in $\mathbb{R}^3$ and parameterized by an arc-length parameter s. The Frenet-Serret bases consist of the orthonormal vector triad $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ (Fig. 1) where the tangent vector to the curve $\boldsymbol{t}$, the normal vector $\boldsymbol{n}$ and the binormal vector $\boldsymbol{b}$ are calculated as

$$\boldsymbol{t} = \frac{d\boldsymbol{r}(s)}{ds}, \quad \boldsymbol{n} = \frac{\frac{d^2\boldsymbol{r}(s)}{ds^2}}{\left\|\frac{d^2\boldsymbol{r}(s)}{ds^2}\right\|}, \quad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} \tag{1}$$

Two intrinsic properties of the curve: the curvature $\kappa(s)$ and the torsion $\tau(s)$ are given as

$$\kappa(s) = \left\|\frac{d^2\boldsymbol{r}(s)}{ds^2}\right\| \quad and \quad \tau(s) = \frac{d\boldsymbol{n}(s)}{ds} \cdot \boldsymbol{b} \tag{2}$$

Finally, the derivatives of the Frenet-Serret bases can be expressed using the Frenet-Serret formula [32], i.e.,

$$\begin{bmatrix} \frac{d\boldsymbol{t}}{ds} \\ \frac{d\boldsymbol{n}}{ds} \\ \frac{d\boldsymbol{b}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{3}$$

Let $\boldsymbol{\varphi} = \varphi_t \boldsymbol{t} + \varphi_n \boldsymbol{n} + \varphi_b \boldsymbol{b}$ be a smooth vector field represented in the Frenet-Serret bases. Using the chain rule, the derivative of this field with respect to the arc-length parameter can be expressed as

$$\frac{d\boldsymbol{\varphi}}{ds} = \left(\frac{d\varphi_t}{ds} - \kappa\varphi_n\right)\boldsymbol{t} + \left(\frac{d\varphi_n}{ds} + \kappa\varphi_t - \tau\varphi_b\right)\boldsymbol{n} + \left(\frac{d\varphi_b}{ds} + \tau\varphi_n\right)\boldsymbol{b} \tag{4}$$

2.2. Equilibrium equations

For a 3-D curved Timoshenko beam expressed in the Frenet-Serret bases, the internal force vector $\boldsymbol{Q}$ and moment vector $\boldsymbol{M}$ are