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A higher order transversely deformable shell-type spectral finite element for dynamic analysis of isotropic structures

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ABSTRACT

This paper deals with certain aspects related to the dynamic behaviour of isotropic shell-like structures analysed by the use of a higher order transversely deformable shell-type spectral finite element newly formulated and the approach known as the Time-domain Spectral Finite Element Method (TD-SFEM). Although recently this spectral approach is reported in the literature as a very powerful numerical tool used to solve various wave propagation problems, its properties make it also very well suited to solve static and dynamic modal problems. The robustness and effectiveness of the current spectral approach has been successfully demonstrated by the authors in the case of thin-walled spherical shell structures through a series of numerical tests comprising the analysis of natural frequencies and modes of vibration of an isotropic spherical shell as well as the wave propagation analysis in the case of the same spherical shell and a half-pipe shell-like structure.

1. Introduction

Investigation, modelling and analysis of wave propagation in shell-like structures have been the subject of scientific interest for many years [1–3]. As a result, during that time various continuous [4–8] and discrete models [9–11] were developed and tested by many authors.

However, the main problem related to continuous models is that they are usually restricted to structures of simple geometries and boundary conditions, as well as homogeneous material properties. In contrast, discrete models can be easily employed to investigate structures of complex geometries and boundary conditions or material properties. Nevertheless, discrete models, in the case of wave propagation problems, need proper spatial discretisation. Among many discrete methods, which are often applied for wave propagation modelling and analysis, the Spectral Finite Element Method (SFEM) appears as an effective and powerful tool [12]. However, it should be remembered that two different spectral approaches exist in the literature. The first is called the Frequency-domain Spectral Finite Element Method (FD-SFEM) and was originally proposed by Doyle [13,14] and later developed by Gopalakrishnan [15,16]. The second approach, proposed by Patera [17], is called the Time-domain Spectral Element Method (TD-SFEM). In the case of two-dimensional (2-D) or three-dimensional problems (3-D) the time-domain formulation of SFEM is much more effective than the frequency-domain formulation of the method.

In fact TD-SFEM is very similar to the well-known Finite Element Method (FEM). The main difference between them comes from the fact that TD-SFEM employs elemental shape functions based on Lobatto or Chebyshev approximation polynomials with elemental nodes located at points, which are the roots of these polynomials. As a consequence the nodes are not equidistant. Additionally, thanks to the orthogonality of the approximation polynomials elemental inertia matrices are diagonal in the case of Lobatto polynomials, or almost diagonal in the case of Chebyshev polynomials. Such forms of elemental inertia matrices allow for the application of more effective and accurate as well as less time consuming techniques to integrate the equations of motion.

In this paper a new multi-mode formulation of a higher order transversely deformable shell-type spectral finite element (SFE) for dynamic analysis of isotropic structures is presented and analysed. The accuracy of the formulation proposed is assessed by comparison of dispersion curves obtained for the current model with dispersion curves obtained for exact solutions of the problem as well as comparison with shell theories well-known from the literature.

Finally, in order to demonstrate the effectiveness of the current formulation of a higher order transversely deformable shell-type SFE a series of numerical tests were performed. These comprised the analysis of natural frequencies and modes of vibration of an isotropic spheri-

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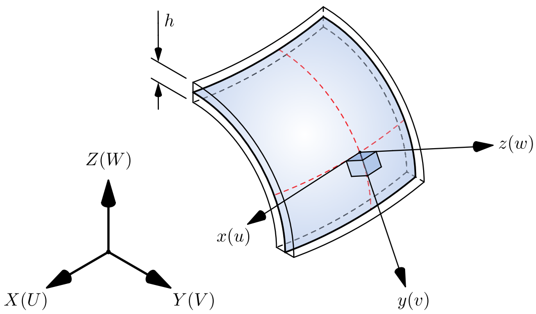

Fig. 1. A shell SFE in the local *xyz* and global *XYZ* coordinate systems.

cal shell as well as the wave propagation analysis in the case of the same spherical shell and a half-pipe shell-like structure. Thanks to this, appropriate conclusions were formulated as a general guidance for the application of the current element and solution techniques in the case of various dynamic problems.

2. Shell element formulation

2.1. Displacement field

In comparison to classical shell theories the current formulation of a higher order transversely deformable shell-type SFE takes advantage of an extended form of the displacement field. According to this formulation the element has six degrees of freedom and takes into account the transverse deformation of the element. This feature of the element becomes very important, when high frequency dynamic responses are to be studied [11]. The shell element under investigation is presented in Fig. 1.

Additional displacement terms of the element displacement field represent higher-order terms of the field expansion into the Maclaurin series. They can be evaluated thanks to the application of the zero-traction boundary conditions for τ_{yz}, τ_{zx} as well as σ_{zz} stress components on the lower and upper surfaces of the element [1–3], in a similar manner as shown in Ref. [18]. Following the same approach as used in Refs. [19,20] the displacement field of the current shell element may be represented, in the local coordinate system of the element *xyz*, as:

$$\begin{aligned} u &= \phi_0 + a\zeta\phi_1 + (1-\zeta^2)\phi_2 + a\zeta(1-\zeta^2)\phi_3 \\ v &= \psi_0 + a\zeta\psi_1 + (1-\zeta^2)\psi_2 + a\zeta(1-\zeta^2)\psi_3 \\ w &= \theta_0 + a\zeta\theta_1 + (1-\zeta^2)\theta_2 + a\zeta(1-\zeta^2)\theta_3 \end{aligned} \tag{1}$$

where symbols ζ and a are defined by relations $z = a\zeta$ and $h = 2a$, while h is the thickness of the element.

It should be noted that the displacement components $\phi_i(i = 0, \ldots, 3)$, $\psi_i(i = 0, \ldots, 3)$ and $\theta_i(i = 0, \ldots, 3)$, remain certain unknown functions of the spatial coordinates x and y defined at the mid-plane of the element. They can be associated with either symmetric (*membrane*) or anti-symmetric (*bending*) behaviour of the element. In the case of the symmetric (*membrane*) behaviour these are the in-plane displacement functions $\phi_i(i = 0, 2)$ and $\psi_i(i = 0, 2)$ as well as the transverse displacement functions $\theta_i(i = 1, 3)$. On the other hand the anti-symmetric (*bending*) behaviour is associated with the remaining in-plane displacement functions $\phi_i(i = 1, 3)$ and $\psi_i(i = 1, 3)$ as well as the transverse displacement functions $\theta_i(i = 0, 2)$.

As mentioned earlier the application of the zero-traction boundary conditions for τ_{yz}, τ_{zx} as well as σ_{zz} stress components on the upper and lower surfaces of the element enables one to reduce the total number of unknown functions (element degrees of freedom) from eight to six. This leads to certain relations for the higher-order terms $\phi_i(i = 2, 3)$, $\psi_i(i = 2, 3)$ and $\theta_i(i = 2, 3)$, which can be expressed as dependent on the remaining lower order terms $\phi_i(i = 0, 1)$, $\psi_i(i = 0, 1)$ and $\theta_i(i = 0, 1)$ for the symmetric and anti-symmetric displacement components:

- for symmetric (*membrane*) behaviour:

$$\begin{aligned} 2\phi_2 &= a^2\partial_x\theta_1 \\ 2\psi_2 &= a^2\partial_y\theta_1 \\ 2\theta_3 &= \theta_1 + \frac{\lambda}{\lambda+2\mu}(\partial_x\phi_0 + \partial_y\psi_0) \end{aligned} \tag{2}$$

- for anti-symmetric (*bending*) behaviour:

$$\begin{aligned} 2\phi_3 &= \phi_1 + \partial_x\theta_0 \\ 2\psi_3 &= \psi_1 + \partial_y\theta_0 \\ 2\theta_2 &= a^2\frac{\lambda}{\lambda+2\mu}(\partial_x\phi_1 + \partial_y\psi_1) \end{aligned} \tag{3}$$

where λ and μ are the Lamé constants.

Taking into account the relations given by Eqs. (2) and (3) the strain field associated with the current definition of the displacement field can be easily defined. Based on that definition the characteristic elemental inertia [$\boldsymbol{M}$] and stiffness [$\boldsymbol{K}$] matrices can be evaluated after assuming a certain polynomial order m as well as an approximation method (Lobatto of Chebyshev) for the unknown functions $\phi_i(i = 0, 1)$, $\psi_i(i = 0, 1)$ and $\theta_i(i = 0, 1)$. This common and standard procedure for the classical FEM approach as well as TD-SFEM is well described and presented in Refs. [21–23].

However, due to the fact that the higher order terms, given by Eqs. (2) and (3), involve local derivatives of the unknown functions $\phi_i(i = 0, 1)$, $\psi_i(i = 0, 1)$ and $\theta_i(i = 0, 1)$, the evaluation process of the characteristic elemental inertia [$\boldsymbol{M}$] and stiffness [$\boldsymbol{K}$] matrices is presented with more details in the following Section 2.4 of this paper.

2.2. Dispersion curves

Dispersion relations or dispersion curves provide very important information about the dependence of the phase and group velocities c_p and c_g on the frequency f, or the wave number k, for elastic waves propagating within structures of interest. They also help to estimate signal propagation times or distances, which on the other hand is very important in all damage detection strategies that are based on the propagation of guided elastic waves [24–27]. Dispersion relations for thin isotropic plates were extensively studied in the past by many researchers, with the results of the fundamental analytical research on that subject presented in Refs. [28,29]. Against these analytical relations all new theories developed should be assessed and verified in order to define their applicability range. This procedure was also used by the authors of this work.

Following the methodology described in Ref. [11] the dispersion curves for the phase c_p and group velocities c_g, related with the displacement field given by Eq. (1), can be obtained in a relatively straightforward manner by the use of Hamilton's principle [14]. Under assumption of small strains and displacements, the virtual work $\mathbb{W}$ associated with the deformation and motion of the shell element under investigation, may be expressed in terms of its strain energy $\mathbb{U}$, kinetic energy $\mathbb{T}$, as well as the work of external forces $\mathbb{F}$. Bearing in mind the relations given by Eqs. (2) and (3) a set of coupled equations of motion can be obtained for the unknown functions $\phi_i(i = 0, 1)$, $\psi_i(i = 0, 1)$ and $\theta_i(i = 0, 1)$ that can be presented in the following form: