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A simple approach for finite element simulation of reinforced plates

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ABSTRACT

We present a new approach for adding Bernoulli beam reinforcements to Kirchhoff plates. The plate is discretised using a continuous/discontinuous finite element method based on standard continuous piecewise polynomial finite element spaces. The beams are discretised by the CutFEM technique of letting the basis functions of the plate represent also the beams which are allowed to pass through the plate elements. This allows for a fast and easy way of assessing where the plate should be supported, for instance, in an optimization loop.

1. Introduction

Reinforcements of plates using lower–dimensional structures such as beams are often employed for the purpose of increasing buckling loads and avoiding eigenfrequencies in vibration problems. The effect of stiffeners can be simulated in a finite element context in a variety of ways. The important issue is how to couple the variables of the beam to the variables of the plate. Different approaches have been suggested:

- Point (nodal) constraints matching beam and plate displacements [20].
- Lagrange multipliers to tie the beam and plate [17].
- Using the plate basis functions also for the beam, along edges or aligned with the elements [5], or obliquely [13,16,19].

The last approach has only been used in the context of Timoshenko beams coupled to Mindlin–Reissner plates, where simple C^0 approximations can be used; a similar approach was recently suggested for modeling embedded trusses by Lé, Legrain, and Moës [15]. In this paper we present a method for the coupling of Kirchhoff plates and Euler–Bernoulli beams based on this concept, together with a tangential differential approach which simplifies the implementation for arbitrarily oriented beams. This is possible thanks to the development of continuous/discontinuous Galerkin (c/dG) methods for higher order problems [6,7,9,10], avoiding the use of C^1–continuity.

The fact that we do not have to employ higher continuity allows for coupling in other contexts as well. In Ref. [12] we proposed to use the same finite element space for the beam as for the higher dimensional structure modeled by linear elasticity, using second order polynomials for elasticity and taking the restriction, or trace, of these polynomials to model the beam using c/dG.

2. Modeling of reinforced plates

2.1. The basic approach

In this Section we develop a simple model of a set of beam elements in a plate. The main approach is as follows:

- Given a continuous finite element space, based on at least second order polynomials for the plate, we define the finite element space for the one–dimensional structure as the restriction of the plate finite element space to the structure which is geometrically modeled by an embedded curve or line.
- To formulate a finite element method on the restricted or trace finite element space we employ continuous/discontinuous Galerkin approximations of the Euler–Bernoulli beam model. The beams are then modeled using the CutFEM paradigm and the stiffness of the embedded beams is in the most basic version, which we consider here, simply added to the plate stiffness.

To ensure coercivity of the cut beam model we in general need to add a certain stabilization term which provides control of the discrete functions variation in the vicinity of the beam. However, for beams embedded in a plate, the plate stabilizes the beam discretizations, and

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we shall show that if the plate is stiff enough compared to the beam the usual additional stabilization [1] is superfluous. The plate problem may also be viewed as an interface problem in order to more accurately approximate the plate in the vicinity of the beam structure; this approach is however significantly more demanding from an implementation point of view and we leave it for future work.

The work presented here is an extension of earlier work [4] where membrane structures were considered, in which case a linear approximation in the bulk suffices.

2.2. The Kirchhoff–Love plate model

In the Kirchhoff–Love plate model, posed on a polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}_{\partial\Omega}$, we seek an out–of–plane (scalar) displacement u to which we associate the strain (curvature) tensor

$$\boldsymbol{\epsilon}(\nabla u) := \frac{1}{2}(\nabla \otimes (\nabla u) + (\nabla u) \otimes \nabla) = \nabla \otimes \nabla u = \nabla^2 u \tag{1}$$

and the plate stress (moment) tensor

$$\boldsymbol{\sigma}_P(\nabla u) := C_P\left(\boldsymbol{\epsilon}(\nabla u) + \nu_\Omega(1-\nu_\Omega)^{-1}\mathrm{div}\,\nabla u\,\boldsymbol{I}\right) \tag{2}$$

$$= C_P\left(\nabla^2 u + \nu_\Omega(1-\nu_\Omega)^{-1}\Delta u\boldsymbol{I}\right) \tag{3}$$

where

$$C_P = \frac{E_\Omega t_\Omega^3}{12(1+\nu_\Omega)} \tag{4}$$

with E_Ω the Young's modulus, ν_Ω the Poisson's ratio, and t_Ω denotes the plate thickness. Since $0 \le \nu_\Omega \le 0.5$ the constants are uniformly bounded.

The Kirchhoff–Love problem then takes the form: given the out–of–plane load (per unit area) f, find the displacement u such that

$$\mathrm{div}\,\mathbf{div}\boldsymbol{\sigma}_P(\nabla u) = f \quad \text{in } \Omega \tag{5}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{6}$$

$$\boldsymbol{n}_{\partial\Omega} \cdot \nabla u = 0 \quad \text{on } \partial\Omega \tag{7}$$

where **div** and div denote the divergence of a tensor and a vector field, respectively.

Weak Form. The variational problem takes the form: Find the displacement $u \in V_\Omega = H_0^2(\Omega)$ such that

$$a_\Omega(u,v) = l_\Omega(v) \quad \forall v \in V_\Omega \tag{8}$$

where the forms are defined by

$$a_\Omega(v,w) = (\boldsymbol{\sigma}_P(\nabla v), \boldsymbol{\epsilon}(\nabla w))_\Omega \tag{9}$$

$$l_\Omega(v) = (f,v)_\Omega \tag{10}$$

We employ the following notation: $L^2(\omega)$ is the Lebesgue space of square integrable functions on ω with scalar product $(\cdot,\cdot)_{L^2(\omega)} = (\cdot,\cdot)_\omega$ and $(\cdot,\cdot)_{L^2(\Omega)} = (\cdot,\cdot)$, and norm $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_\omega$ and $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$, $H^s(\omega)$ is the Sobolev space of order s on ω with norm $\|\cdot\|_{H^s(\omega)}$, and $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega\}$, and $H_0^2(\Omega) = \{v \in H^2(\Omega) : v = \boldsymbol{n}_{\partial\Omega} \cdot \nabla v = 0 \text{ on } \partial\Omega\}$.

2.3. The Euler–Bernoulli beam model

Consider a straight thin beam with centerline $\Sigma \subset \Omega$ and a rectangular cross-section with width b_Σ and thickness t_Σ, see Fig. 2. The modeling of the beam is performed using tangential differential calculus and we follow the exposition in Refs. [11,12], which also covers curved beams. Using this approach the beam equation is expressed in the same coordinate system as the plate, which is convenient in the construction of the cut finite element method for reinforced plates, see Fig. 1 for examples.

Let $\boldsymbol{t}$ be the tangent vector to the line Σ and $\boldsymbol{P}_\Sigma = \boldsymbol{t} \otimes \boldsymbol{t}$ the projection onto the one dimensional tangent space of Σ and define the tangential derivatives

$$\nabla_\Sigma v = \boldsymbol{P}_\Sigma \nabla v, \quad \partial_t v = \boldsymbol{t} \cdot \nabla v \tag{11}$$

Then we have the identity

$$\nabla_\Sigma v = (\partial_t v)\boldsymbol{t} \tag{12}$$

Based on the assumption that planar cross sections orthogonal to the midline remain plane after deformation we assume that the displacement takes the form

$$\boldsymbol{u} = u\boldsymbol{n} + \theta\zeta\boldsymbol{t} \tag{13}$$

where ζ is the signed orthogonal distance to Ω, positive on one side of Ω and negative on the other side, and $\theta : \Sigma \to \mathbb{R}$ is an angle representing an infinitesimal rotation, assumed constant in the normal plane.

In Euler–Bernoulli beam theory the beam cross-section is assumed plane and orthogonal to the beam midline after deformation and no shear deformations occur, which means that we have

$$\theta = \boldsymbol{t} \cdot \nabla u := \partial_t u \tag{14}$$

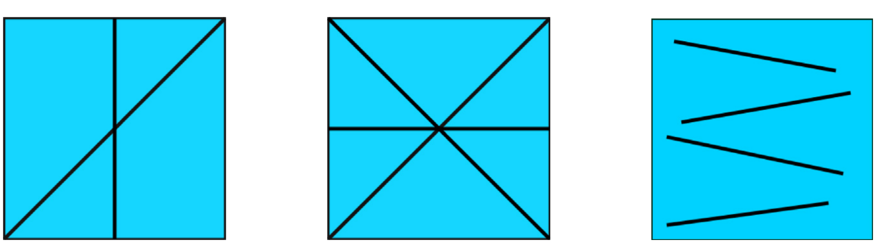

Fig. 1. Examples of plates reinforced by beams.

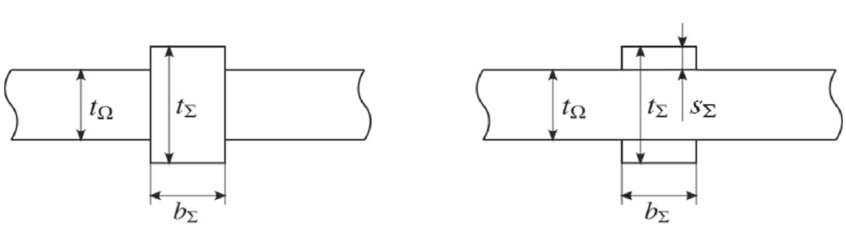

Fig. 2. Left: The reinforced plate geometry parameters, t_Ω, t_Σ, and b_Σ. Right: Alternative design of reinforcement with two separate beams of thickness $s_\Sigma = (t_\Sigma - t_\Omega)/2$ above and below the plate.