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A novel weak form three-dimensional quadrature element solution for vibrations of elastic solids with different boundary conditions

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ABSTRACT

Three-dimensional (3D) vibration behavior of elastic parallelepipeds, including beams, plates, and solids, is critical for a wide range of engineering applications. However, obtaining accurate 3D solutions of parallelepipeds is a relatively challenging task. In this paper, a novel and general 3D weak form quadrature element method (QEM) is presented for solutions of vibrations of parallelepipeds with different combinations of boundary conditions. The element stiffness and mass matrices are explicitly derived via the numerical integration together with the differential quadrature (DQ) law. A number of case studies on beams, thin and thick plates, and 3D solids with different combinations of boundary conditions have been conducted. The natural frequencies and mode shapes were in excellent agreement with existing results and data obtained by the finite element method with a very fine mesh. It is seen that the proposed 3D quadrature element is simple in formulations, computationally efficient and capable of capturing the 3D vibration behavior of parallelepipeds with high precision. In addition, some new frequencies and mode shapes are provided to augment the archived reference frequencies and mode shapes.

1. Introduction

Beams, plates and shells are the basic structural elements in engineering applications. Their vibration behavior is of important to the designers and engineers and thus has been received great attentions. With a choice of different parameters, parallelepipeds can represent a number of structural elements, such as beams, thin and thick plates, and solids. Therefore, three-dimensional (3D) vibration behavior of elastic parallelepipeds with different combinations of boundary conditions is important for a wide range of engineering applications [1]. Accurate 3D vibration solutions of parallelepipeds can be used not only for practical applications but also for evaluations of the precision of various lower-order beam and plate theories [2–6]. However, obtaining accurate 3D vibration solutions of parallelepipeds is a relatively challenging task, and thus much fewer works as compared to the ones of beams, plates [7] and shells [8] have been reported thus far.

In the literature, several approaches have been used for solving the 3D vibration problems of elastic structures with parallelepiped shape, including the finite element method (FEM), the finite difference method (FDM), various Ritz methods, and the spectral method (SM) [1–6]. The methods of finite element and finite difference are versatile approaches, but they are computationally inefficient. The 3D spectral method such as the 3D spectral-Tchebychev method [1] is computationally more efficient than the FDM and FEM, since the SM possesses exponential rate of convergence. The rate of convergence of various Ritz methods depends not only on the assumed displacement functions but also on the boundary conditions [2–6].

It is seen that only frequencies and mode shapes of doubly symmetry (SS), symmetry-antisymmetry (SA), antisymmetry-symmetry (AS), and doubly antisymmetry (AA) about two orthogonal planes of parallelepipeds, i.e., about the $x = a/2$ and $y = b/2$ planes shown in Fig. 1, have been reported in literature [1–6]. Since the parallelepiped possesses three symmetric planes, i.e., the $x = a/2$, $y = b/2$ and $z = c/2$ planes, other coupled mode shapes also exist for the 3D vibrating parallelepipeds with different combinations of boundary conditions and may be important in practical applications.

Currently several other simple and efficient numerical methods are available and may be employed for obtaining accurate 3D vibration solutions of parallelepipeds, such as the differential quadrature method (DQM) [9–13], the discrete singular convolution (DSC) algorithm [14–23], and the weak form quadrature element method (WQEM or simply QEM) [24–30].

Both DQM and DSC belong to the strong form methods. The DQM [9–13] is simple and can obtain accurate solutions with minimum

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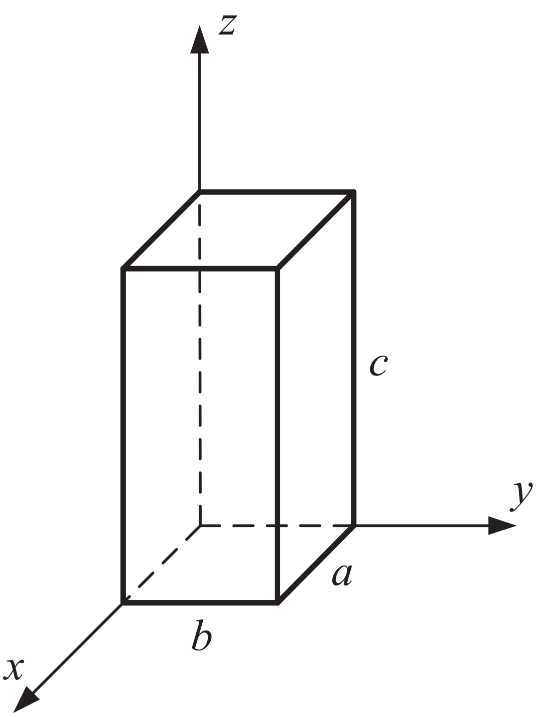

Fig. 1. Sketch of an elastic parallelepiped.

number of grid points. The DSC, proposed by Wei [14–17], possesses an excellent feature of obtaining relatively accurate high mode frequencies at the same time [18,19,22] besides obtaining accurate lower mode frequencies. Being a high order finite element method, the QEM [24–30] possesses advantages of the flexibility of the FEM and high accuracy of the DQM.

From literature reviews, however, the DQM, the DSC, and the QEM have been used only for one- or two-dimensional vibration analysis of elastic beams, plates and shells. Perhaps one of the possible reasons is that the programming for 3D analysis by using these methods poses some challenges.

In this paper, the QEM is used for solving the problem of free vibrations of 3D elastic solids with different boundary conditions. The reasons to select the QEM are that the method possess exponential rate of convergence, its formulation is simpler than the existing one of the 3D spectral-Tchebychev method, the implementation of various boundary condition is simple, and the programming effort is less than the DQM and the DSC. For the strong form methods, such as the DQM and the DSC, difficulties may arise in the implementation of the free boundary conditions, especially at the free corner point. There are six zero stress conditions, but only three of them can be applied by the strong form methods, such as the DQM and the DSC. Although the iteratively matched boundary method [20], matched interface and boundary method [21], and Taylor series expansion method [22] can be used to apply the free boundary conditions, but these methods are not as convenient as the ones of the weak form methods, such as the QEM and FEM. In the QEM and FEM, the zero stress conditions are natural boundary conditions, and thus are satisfied automatically. In other words, the C_0 continuity is enforced. Moreover, it is also possible to develop weak form quadrature element satisfying higher order continuity requirements, i.e., the C_1 continuity is enforced [8].

The weak form quadrature element method is originally proposed by Striz et al. [24,25]. The QEM is different from the strong form differential quadrature element method presented in Refs. [10,11], since its formulations are based on the minimum potential energy. In principle, the QEM can be regarded as a high order finite element method. The major differences from the conventional high-order FEM are (a) Element nodes are distributed non-uniformly; and (b) Explicit formulations of element stiffness and mass matrices are obtained by numerical integration via the help of the differential quadrature law. Therefore, the element can be implemented adaptively, i.e., the number of element nodes can be changed arbitrarily according to the requirement of the solution accuracy. It has been demonstrated in solving one-dimensional (1D) and two-dimensional (2D) problems that the QEM is computationally efficient [30] and also possess exponential rate of convergence [24,25].

It should be mentioned that the research groups of Zhong [26,27], Xing [28] and Wang [29] have made important contributions to the development of the QEM. Now any kind of points can be used as the element nodes and both Gauss quadrature and Gauss–Lobatto–Legendre (GLL) quadrature can be employed in the numerical integration. Although the QEM is frequently regarded as the spectral element method (SEM) [31,32], popular in the wave field due to its exponential rate of convergence. However, differences between the QEM and the SEM [33–38] do exist and are clearly given in Refs. [24,25,30].

The objective of present investigation is to present a new solution technique, i.e., the QEM, for analyzing the 3D vibration behavior of elastic parallelepipeds efficiently. A general 3D quadrature parallelepiped element is proposed. The novelties of the proposed QEM are that any types of points can be used as element nodes and that Gauss quadrature can be used in numerical integration. Therefore, the method is more general than the existing QEM and the SEM.

Detailed formulations are given. A number of case studies on beams, plates, and solid cubes with different combinations of boundary conditions are conducted by using the proposed 3D quadrature parallelepiped element. For verifications, numerical results are compared with existing solutions and data obtained by the finite element method with a fine mesh. In addition, some new frequencies and modes shapes are provided to augment the archived reference frequencies and mode shapes. Finally, conclusions are drawn based on the results reported in this paper.

2. Novel weak form 3D quadrature element formulations

2.1. Expressions of strain energy and kinetic energy

Let symbols V, E, G, μ, ρ represent the volume of the elastic solid, elasticity modulus, shear modulus, Poisson's ratio, and the mass density, respectively. The strain energy of an isotropic homogenous elastic solid can be symbolically written as

$$U = \frac{1}{2}\iiint_V [\{\varepsilon\}^T[C]\{\varepsilon\} + \{\gamma\}^T[G]\{\gamma\}]dV \tag{1}$$

where $\{\varepsilon\}$ and $\{\gamma\}$ are the normal strain and shear strain vectors defined by

$$\{\varepsilon\}^T = \left\lfloor \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial w}{\partial z} \right\rfloor, \ \{\gamma\}^T = \left\lfloor \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\rfloor \tag{2}$$

and $[C]$ and $[G]$ are 3×3 stiffness matrices of the material defined by

$$[C] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu \\ \mu & 1-\mu & \mu \\ \mu & \mu & 1-\mu \end{bmatrix}, \quad [G] = G\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

where u, v and w are displacement components in x, y and z directions, and x-y-z is the Cartesian coordinate system shown in Fig. 1.

The kinetic energy of an isotropic homogenous elastic solid is given by

$$T = \frac{\rho}{2}\iiint_V \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]dV \tag{4}$$

where t is the time.