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Three-dimensional hybrid-mixed stress elements for free vibration analysis

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ABSTRACT

Hexahedral hybrid-mixed finite elements are proposed for free vibration analysis of three-dimensional solids. The element formulation relies on the simultaneous and independent approximations of stress and displacement in the element domain as well as the displacement on their boundary. Sets of complete and linearly independent non-nodal Legendre polynomials used for the field variables lead to symmetric, highly sparse and well conditioned solving systems. Numerical tests show that the elements yield accurate results, even in the presence of high stress gradients, and they seem to be free of shear and volumetric locking and have low sensitivity to mesh distortion. The hierarchical *p*-refinement strategy is exploited.

1. Introduction

Mixed, hybrid and hybrid-mixed finite element formulations have been developed to offer some advantages compared to the standard displacement-based formulation [1]. In particular, the hybrid-mixed stress formulation is based on the simultaneous and independent approximations of stress and displacement within the element and displacement on its boundary. All the field equations, interelement continuity and boundary conditions are enforced on average. The formulation, as first conceived by Freitas et al. [2] for non-nodal basis functions, is quite general in the sense that any complete and linearly independent set of functions is a potential candidate for stress and displacement bases because no explicit constraints are placed on them.

Freitas and Bussamra [3] have proposed hybrid-Trefftz stress elements for the static analysis of three-dimensional isotropic solids. The stress approximation bases are extracted from the Papkovitch-Neuber solution of Navier equation, assigning Legendre and Chebyshev polynomials to Papkovitch-Neuber potentials. Displacements on the element boundary are approximated by independent hierarchical monomial bases. These elements have shown no locking, low sensitivity to geometric irregularities and good estimates for stresses and displacements.

Nonconventional finite elements have been developed for the elastostatic analysis in recent years. Bussamra et al. [4] have modified the formulation in Ref. [3] to alleviate the constraints on the stress approximation so that only the equilibrium equation should be satisfied locally. The basis functions used in Ref. [3] are then adopted to develop hybrid stress elements for three-dimensional static analysis of laminated composite plates. Two-dimensional hybrid-Trefftz elements for thick orthotropic plates were presented by Karkon et al. [5]. Resaiee-Pajand and Karkon [6,7] addressed hybrid-Trefftz models for thin plate bending. Hybrid-Trefftz elements for Reissner-Mindlin isotropic plates were presented by Pereira and Freitas [8]. Hybrid-mixed models for buckling analysis of plates and framed structures were developed by Arruda et al. [9]. Santos et al. [10] proposed a hybrid-mixed finite element formulation for three-dimensional framed structures under strong geometric nonlinearity.

Elastoplastic analysis by 3D nonconventional finite elements have been proposed by Bussamra et al. [11], using hybrid-Trefftz elements, and Mendes and Castro [12], using hybrid-mixed stress elements.

Hybrid-Trefftz and hybrid-mixed elements have been proposed for damage analysis in two-dimensional domains by Argôlo and Proença [13] and Souza and Proença [14] and in three-dimensional domains by Bussamra et al. [15] and Góis and Proença [16].

Besides isotropic and orthotropic materials, hybrid-Trefftz and hybrid-mixed elements have also been applied to the analysis of heterogeneous composites [17], concrete [18,19], unsaturated porous media [20], biphasic media [21] and hydroxypropyl cellulose structures [22].

In general, hybrid, hybrid-Trefftz and hybrid-mixed elements are based on the use of polynomials, such as Legendre or Chebyshev, for the approximations of stress and displacements. Castro et al. [23–25] used wavelets instead.

Hybrid and hybrid-Trefftz stress formulations find difficulties to be extended to dynamics problems due to the presence of unknown

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displacements in the equation of motion. Pereira and Freitas [26] presented a set of hybrid finite element formulations for structural dynamics, namely the hybrid-mixed, hybrid and hybrid-Trefftz, and applied to rods and plate stretching to determine natural frequencies and forced motions. Freitas [27] established an incremental procedure for the solution of hyperbolic problems based on mixed finite elements. A mixed finite element formulation has also been applied to the analysis of Reissner-Mindlin plates [28] and shells of revolution [29].

In this paper, the formulation in Ref. [4] is converted into a hybrid-mixed stress formulation by further alleviating the stress approximation of satisfying locally the equilibrium equation, now replaced by the equation of motion, and used to generate three-dimensional elements for free vibration analysis. All the approximations are carried out with sets of complete and linearly independent non-nodal Legendre polynomials in view of their attractive properties described in Refs. [30,31]. The solving system obtained is highly sparse with coefficients established in closed form to avoid numerical integration. The primary objective of this work is to supplement the existing literature on hybrid-mixed stress finite element formulation with 3D nonconventional elements applied to elastodynamic problems.

Three-dimensional hybrid-Trefftz and hybrid-mixed finite elements generate large systems particularly in *p*-refinement, a potential problem in memory storage and processing time. The high sparsity of the resulting matrices of the proposed elements enables the use of numerical routines that deal with large sparse matrices. These routines save computer memory by storing only non-null elements, and save computing time by reducing the number of numerical calculations [32].

The idea of reducing the number of elements and hence the mesh size, as incorporated in the hybrid-mixed stress formulation herein, is important. Meshing of 3D elements is still one of the most burdensome tasks in finite element analysis. Mesh generation takes a noticeable part of the user time, especially in problems with complex geometries and meshes of hexahedral elements [33]. Reduction of mesh size is also important in the analysis cost because it reduces time with element assemblage.

Spurious stress modes are avoided in the newly developed elements as a consequence of the approximation by sets of linearly independent functions. Spurious displacement modes are in turn avoided by establishing appropriate relations on the degrees of the stress and displacement polynomials used in their approximations. The completeness and hierarchical nature of the Legendre polynomials suggest the use of coarse meshes enhanced by *p*-refinement. The element performance is illustrated by means of free vibrations problems, where good convergence rates and accurate estimates of natural frequencies and mode shapes have been observed. The elements seem to be free of shear and volumetric locking and have low sensitivity to mesh distortion.

2. Finite element formulation

Let V be the domain of a solid and Γ its boundary. Let Γ_u and Γ_σ be the portions of Γ on which displacements and tractions are specified, respectively. In free vibration problems, the assumed harmonic motion with circular frequency ω reduces the equation of motion and strain-displacement relation to

$$\mathbf{D}\boldsymbol{\sigma} + \rho\omega^2\mathbf{u} = 0 \quad \text{in V} \quad \boldsymbol{\varepsilon} = \mathbf{D}^T\mathbf{u} \quad \text{in V} \tag{1}$$

for a material with density ρ. Vector $\mathbf{u}$ represents the displacement amplitude, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are vectors which collect the amplitudes of the independent components of stress and strain tensors and $\mathbf{D}$ is a differential operator. On the boundary,

$$\mathbf{N}\boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_\sigma \quad \mathbf{u} = 0 \quad \text{in } \Gamma_u. \tag{2}$$

Matrix $\mathbf{N}$ contains the components of the unit outward normal vector to Γ; the specified surface tractions on Γ_σ and displacements on Γ_u are assumed to be null. The strain-stress relation of a linearly hyperelastic material are denoted by

$$\boldsymbol{\varepsilon} = \mathbf{f}\boldsymbol{\sigma} \quad \text{in V} \tag{3}$$

where $\mathbf{f}$ is the 6×6 material compliance matrix.

Suppose that the solid is divided into a number of elements and treated as an assembly of them. To apply the above equations to a typical element, the definition of the boundary should be extended to include the traction and displacement continuity on the interelement portion $\Gamma_i(\Gamma_u \cup \Gamma_\sigma \cup \Gamma_i = \Gamma;\ \Gamma_u \cap \Gamma_\sigma \cap \Gamma_i = \emptyset)$:

$$(\mathbf{N}\boldsymbol{\sigma})^+ + (\mathbf{N}\boldsymbol{\sigma})^- = 0 \quad \text{on } \Gamma_i \quad \mathbf{u}^+ = \mathbf{u}^- \quad \text{in } \Gamma_i \tag{4}$$

where the superscripts "+" and "−" denote the two sides of Γ_i.

The hybrid-mixed elements are based on the simultaneous and independent approximations of stress $\boldsymbol{\sigma}$ and displacement $\mathbf{u}$ in V, and displacement $\tilde{\mathbf{u}}$ on $\Gamma_\sigma,\ \Gamma_i$:

$$\boldsymbol{\sigma} = \mathbf{S}\mathbf{X} \quad \text{in V} \quad \mathbf{u} = \mathbf{U}_\text{v}\mathbf{q}_\text{v} \quad \text{in V} \qquad \tilde{\mathbf{u}} = \mathbf{U}_\Gamma\mathbf{q}_\Gamma \quad \text{on } \Gamma_\sigma,\ \Gamma_i. \tag{5}$$

Matrices $\mathbf{S}$, $\mathbf{U}_\text{v}$ and $\mathbf{U}_\Gamma$ collect the approximation functions and vectors $\mathbf{X}$, $\mathbf{q}_\text{v}$ and $\mathbf{q}_\Gamma$ collect unknown coefficients.

2.1. Weak form of equations

The first equations in (1), (2) and (4) can be enforced to be satisfied in a weak sense according to

$$\int_\text{V} \delta\mathbf{u}^T\left(\mathbf{D}\boldsymbol{\sigma} + \rho\omega^2\mathbf{u}\right)\text{dV} - \int_{\Gamma_\sigma} \delta\tilde{\mathbf{u}}^T\mathbf{N}\boldsymbol{\sigma}\ \text{d}\Gamma - \int_{\Gamma_i} \delta\tilde{\mathbf{u}}^T\mathbf{N}\boldsymbol{\sigma}\ \text{d}\Gamma = 0. \tag{6}$$

The last integral, when considered jointly with those of the neighborhood elements, enforces the first of conditions (4). The second equations in (1), (2), (4) and equation (3) can also be enforced to be satisfied in a weak sense by means of

$$\begin{aligned}\int_\text{V} \delta\boldsymbol{\sigma}^T(\boldsymbol{\varepsilon} - \mathbf{D}^T\mathbf{u})\ \text{dV} - \int_\text{V} \delta\boldsymbol{\sigma}^T(\boldsymbol{\varepsilon} - \mathbf{f}\boldsymbol{\sigma})\ \text{dV} + \int_{\Gamma_u} (\mathbf{N}\,\delta\boldsymbol{\sigma})^T\mathbf{u}\ \text{d}\Gamma \\ + \int_{\Gamma_\sigma\cup\Gamma_i} (\mathbf{N}\,\delta\boldsymbol{\sigma})^T(\mathbf{u} - \tilde{\mathbf{u}})\ \text{d}\Gamma = 0.\end{aligned} \tag{7}$$

The last integral, when evaluated on Γ_i, enforces the second of conditions (4) by constraining the approximation $\tilde{\mathbf{u}}$ to be the same along the common boundaries of any two adjacent elements.

In view of the divergence theorem

$$\int_\text{V} \delta\boldsymbol{\sigma}^T(\mathbf{D}^T\mathbf{u})\ \text{dV} = \int_\Gamma (\mathbf{N}\,\delta\boldsymbol{\sigma})^T\mathbf{u}\ \text{d}\Gamma - \int_\text{V} (\mathbf{D}\,\delta\boldsymbol{\sigma})^T\mathbf{u}\ \text{dV} \tag{8}$$

and some simplifications, expressions (6) and (7) reduce to

$$\begin{aligned}&\int_\text{V} \delta\boldsymbol{\sigma}^T\left(\mathbf{D}\sigma + \rho\omega^2\mathbf{u}\right)\ \text{dV} - \int_{\Gamma_\sigma\cup\Gamma_i} \delta\tilde{\mathbf{u}}^T\mathbf{N}\sigma\ \text{d}\Gamma = 0 \\ &\int_\text{V} \delta\boldsymbol{\sigma}^T\mathbf{f}\boldsymbol{\sigma}\ \text{dV} + \int_\text{V} (\mathbf{D}\,\delta\boldsymbol{\sigma})^T\mathbf{u}\ \text{dV} - \int_{\Gamma_\sigma\cup\Gamma_i} (\mathbf{N}\,\delta\boldsymbol{\sigma})^T\tilde{\mathbf{u}}\ \text{d}\Gamma = 0.\end{aligned} \tag{9}$$

2.2. Element matrices

Substitution of (5) into (9) yields the symmetric system of discrete equations

$$\left(\begin{bmatrix} \mathbf{F} & \mathbf{A} & \mathbf{B} \\ \mathbf{A}^\mathbf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} + \omega^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{bmatrix}\right) \begin{Bmatrix} \mathbf{X} \\ \boldsymbol{q}_\Gamma \\ \boldsymbol{q}_\text{V} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{10}$$