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Mean-strain 8-node hexahedron with optimized energy-sampling stabilization

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ABSTRACT

A method for stabilizing the mean-strain hexahedron was described by Krysl (in IJNME 2014). The technique relied on a sampling of the stabilization energy using the mean-strain quadrature and the full Gaussian integration rule, which was shown to guarantee consistency and stability. The stabilization energy was assumed to be generated by a modified constitutive matrix based on the spectral decomposition. The stabilization required user-selected values of the stabilization parameters. In the present work we eliminate the arbitrariness of the stabilization parameters. We formulate the technique more precisely as an assumed-strain method, and we express the stabilization energy in terms of input parameters of the real material. Finally, we fix the value of the stabilization parameters in a quasi-optimal manner by linking the stabilization to the bending behavior of the hexahedral element. For simplicity the developments are limited to linear elasticity, but with an arbitrarily anisotropic elasticity matrix. The accuracy and convergence characteristics of the present formulations compare favorably with the capabilities of mean-strain and other high-performance hexahedral elements as implemented in Abaqus and with a number of successful hexahedral and shell elements and we demonstrate that the present element performs very well when used with large aspect ratios for thin structures such as plates or shells.

1. Introduction

First-order bricks (hexahedra) tend to be exploited for 3-D analysis for their efficiency, robustness, and ease-of-use. Eight-node mean-strain hexahedra seem to be capable of providing both coarse-mesh accuracy and reliable convergence, and significant progress has been made over the years in this technology.

We refer to [1,2] for overview of the recent developments. The main issues for the mean-strain hexahedra are how to achieve at the same time (i) locking-free response, (ii) good coarse-mesh accuracy, and (iii) stability. Strictly mean-strain hexahedra achieve locking-free response, but lose stability. Adding stability, for instance by treating the so-called hourglassing modes, would tend to affect locking and accuracy. Coarse-mesh accuracy requires of the stabilization to not to deteriorate the response of the element but rather to enhance the ability of the element to accurately respond to deformations in the hourglassing modes such as bending or torsion. Puso's hexahedron is a good example of a successful approach [3].

A method for stabilizing the mean-strain hexahedron that differed from the then-known approaches was described by Krysl [4]. The technique relied on a sampling of the stabilization energy using two quadrature rules, the mean-strain quadrature and the full Gaussian integration rule. The use of two quadrature rules was shown to guarantee both consistency and elimination of the hourglassing modes. The stabilization energy was assumed to be generated by a modified constitutive matrix based on the spectral decomposition. The stabilization required user-selected values of stabilization parameters, which is in general undesirable.

In the present work we eliminate the arbitrariness of the stabilization parameters. Firstly, in Section 2 we formulate the technique more precisely as an assumed-strain method. The stabilization energy is then introduced in Section 3 as a quadratic form which is added and subtracted at the same time: added for strains linked to the displacements and subtracted for the assumed strains. We develop an argument for the resulting hexahedral element being convergent by establishing consistency and positive-semi-definiteness of the strain energy. In Section 4 the parameters of the stabilization material are expressed in terms of input parameters of the real material in a way that avoids locking due to volumetric and other constraints (such as for strongly anisotropic materials). The value of the remaining stabilization parameter (Young's modulus) is fixed in a quasi-optimal manner by linking the stabilization to the bending behavior of the hexahedral element. For simplicity the developments are limited to linear elasticity, but with an arbitrarily anisotropic elasticity matrix.

Section 5 illustrates the performance of the proposed approaches on a variety of benchmark problems, for isotropic and anisotropic

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material models. Importantly, the coarse-mesh response is significantly improved by the choice of the stabilization parameter. This proves important especially for thin shells and plates, where the present element is shown to match the performance of specialized shell and plate elements. The performance of the stabilization is also tested for highly distorted elements in a vibration problem. The present stabilization technique is also shown to work for anisotropic materials.

The accuracy and convergence characteristics of the present formulations compare favorably with the capabilities of mean-strain and other high-performance hexahedral elements as implemented in Abaqus. In addition, we compare with a number of successful hexahedral and shell elements and we demonstrate that the present element performs very well for thin structures such as plates or shells. Crucially, the hexahedron formulation presented here eliminates the need for user-selected values of the stabilization parameters. Together with excellent performance this makes the present element a good general-purpose hexahedron.

2. Assumed-strain formulation

We will re-derive the mean-strain eight-node hexahedra [5,6] taking the variational approach. This will make the inclusion of the stabilization particularly illuminating. We shall take as the starting point the strain-displacement (de Veubeke) functional

$$\Pi(\overline{\boldsymbol{\epsilon}},\boldsymbol{u}) = \Psi(\overline{\boldsymbol{\epsilon}}) + \int_\Omega \overline{\boldsymbol{\sigma}}\cdot(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}})\,d\Omega - \mathcal{W}. \tag{1}$$

Here

$$\Psi(\overline{\boldsymbol{\epsilon}}) = \int_\Omega \mathcal{U}(\overline{\boldsymbol{\epsilon}})\,d\Omega, \tag{2}$$

is the strain energy, where the energy density $\mathcal{U}(\overline{\boldsymbol{\epsilon}})$ is generated by the assumed strains $\overline{\boldsymbol{\epsilon}}$. For instance in linear elasticity we define a quadratic form of the assumed strains $\overline{\boldsymbol{\epsilon}}$

$$\mathcal{U}(\overline{\boldsymbol{\epsilon}}) = \tfrac{1}{2}\overline{\boldsymbol{\epsilon}}^T\mathbf{D}\overline{\boldsymbol{\epsilon}}, \tag{3}$$

with a symmetric positive definite constitutive matrix $\mathbf{D}$. The strains are represented here $\boldsymbol{\epsilon} = [\epsilon_{11},\epsilon_{22},\epsilon_{33},2\epsilon_{32},2\epsilon_{31},2\epsilon_{12}]^T$ in the Voigt–Mandel representation of the second order strain tensor (and analogously for $\overline{\boldsymbol{\epsilon}}$).

We shall assume that the material properties $\mathbf{D}$ and the assumed strains $\overline{\boldsymbol{\epsilon}}$ within each hexahedral element are uniform. The stress $\overline{\boldsymbol{\sigma}}$ is obtained as

$$\overline{\boldsymbol{\sigma}} = \frac{\partial\mathcal{U}(\overline{\boldsymbol{\epsilon}})}{\partial\overline{\boldsymbol{\epsilon}}}. \tag{4}$$

Now we shall express the first variation of the functional (1) as

$$\delta\Pi(\overline{\boldsymbol{\epsilon}},\boldsymbol{u}) = \int_\Omega\left[\frac{\partial\mathcal{U}(\overline{\boldsymbol{\epsilon}})}{\partial\overline{\boldsymbol{\epsilon}}}\cdot\delta\overline{\boldsymbol{\epsilon}} + \delta\overline{\boldsymbol{\sigma}}\cdot(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}}) + \overline{\boldsymbol{\sigma}}\cdot(\delta\boldsymbol{\epsilon}-\delta\overline{\boldsymbol{\epsilon}})\right]d\Omega - \delta\mathcal{W}, \tag{5}$$

where the variation of the stress may be expressed directly as

$$\delta\overline{\boldsymbol{\sigma}} = \frac{\partial^2\mathcal{U}(\overline{\boldsymbol{\epsilon}})}{\partial\overline{\boldsymbol{\epsilon}}^2}\cdot\delta\overline{\boldsymbol{\epsilon}} \tag{6}$$

The matrix of tangent moduli may be identified as

$$\mathbf{D} = \frac{\partial^2\mathcal{U}(\overline{\boldsymbol{\epsilon}})}{\partial\overline{\boldsymbol{\epsilon}}^2}. \tag{7}$$

As usual, the solution to (5) will follow from

$$\delta\Pi(\overline{\boldsymbol{\epsilon}},\boldsymbol{u}) = 0, \tag{8}$$

but we will achieve this by separating out the second and third term that link the displacement-related strains and the assumed strains so that we will require the satisfaction of the balance equation

$$\int_\Omega \frac{\partial\mathcal{U}(\overline{\boldsymbol{\epsilon}})}{\partial\overline{\boldsymbol{\epsilon}}}\cdot\delta\overline{\boldsymbol{\epsilon}}\,d\Omega - \delta\mathcal{W} = 0, \tag{9}$$

and, separately, the satisfaction of the kinematic equation

$$\int_\Omega \delta\overline{\boldsymbol{\epsilon}}\cdot\mathbf{D}\cdot(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}}) + \overline{\boldsymbol{\sigma}}\cdot(\delta\boldsymbol{\epsilon}-\delta\overline{\boldsymbol{\epsilon}})\,d\Omega = 0. \tag{10}$$

The form of the assumed strains will be derived from (10).

At this point we introduce the finite element approximation. Expression (10) is evaluated by adding contributions from individual finite elements. Therefore (10) may be specialized to the domain of a single hexahedral finite element $\Omega^{(e)}$ as

$$\int_{\Omega^{(e)}} \delta\overline{\boldsymbol{\epsilon}}\cdot\mathbf{D}\cdot(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}}) + \overline{\boldsymbol{\sigma}}\cdot(\delta\boldsymbol{\epsilon}-\delta\overline{\boldsymbol{\epsilon}})\,d\Omega = 0. \tag{11}$$

Here we express the strain vectors in dependence on the displacement in the form

$$\boldsymbol{\epsilon} = \mathbf{B}\boldsymbol{U} \quad\text{and}\quad \overline{\boldsymbol{\epsilon}} = \overline{\mathbf{B}}\boldsymbol{U} \tag{12}$$

where the strain-displacement matrix $\mathbf{B}$ is the usual expression calculated from the gradients of the basis functions, the 24×1 vector $\boldsymbol{U}$ consists of the nodal displacements,

$$\boldsymbol{U} = [\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_8]^T \tag{13}$$

and the assumed strain-displacement matrix $\overline{\mathbf{B}}$ will be derived below.

Substitution of (12) into (11) leads to

$$\int_{\Omega^{(e)}} (\mathbf{D}\overline{\mathbf{B}}\delta\boldsymbol{U})\cdot(\mathbf{B}-\overline{\mathbf{B}})\boldsymbol{U} + \overline{\boldsymbol{\sigma}}\cdot(\mathbf{B}-\overline{\mathbf{B}})\delta\boldsymbol{U}\,d\Omega = 0. \tag{14}$$

Extricating the nodal displacement vectors from underneath the integral signs we obtain

$$\delta\boldsymbol{U}\cdot\int_{\Omega^{(e)}}(\mathbf{D}\overline{\mathbf{B}})\cdot(\mathbf{B}-\overline{\mathbf{B}})\,d\Omega\cdot\boldsymbol{U} + \int_{\Omega^{(e)}}\overline{\boldsymbol{\sigma}}\cdot(\mathbf{B}-\overline{\mathbf{B}})\,d\Omega\cdot\delta\boldsymbol{U} = 0. \tag{15}$$

If we recollect now that $\overline{\boldsymbol{\sigma}}$, $\mathbf{D}$, and $\overline{\mathbf{B}}$ are uniform over the domain of the element, we are immediately led for each element e to the condition

$$\delta\boldsymbol{U}\cdot\mathbf{D}\cdot\overline{\mathbf{B}}\cdot\int_{\Omega^{(e)}}(\mathbf{B}-\overline{\mathbf{B}})\,d\Omega\cdot\boldsymbol{U} + \overline{\boldsymbol{\sigma}}\cdot\int_{\Omega^{(e)}}(\mathbf{B}-\overline{\mathbf{B}})\,d\Omega\cdot\delta\boldsymbol{U} = 0, \tag{16}$$

which may be satisfied by taking

$$\int_{\Omega^{(e)}}(\mathbf{B}-\overline{\mathbf{B}})\,d\Omega = \mathbf{0}. \tag{17}$$

Thus we are then in a position to define the assumed strain-displacement matrix as

$$\overline{\mathbf{B}} = \left(\int_{\Omega^{(e)}} d\Omega\right)^{-1}\int_{\Omega^{(e)}}\mathbf{B}\,d\Omega = V_{\mathrm{e}}^{-1}\int_{\Omega^{(e)}}\mathbf{B}\,d\Omega. \tag{18}$$

Here V_{e} is the element volume. The assumed strain-displacement matrix is the mean of the strain-displacement matrix $\mathbf{B}$ over the domain of the element e, hence the designation of the formulation as the mean-strain approach.

The strain energy function (2) modified by the steps above thus reads for a single element

$$\overline{\Psi}_e(\mathbf{U}) = \int_\Omega \mathcal{U}(\overline{\boldsymbol{\epsilon}})\,d\Omega = V_e\mathcal{U}(\overline{\boldsymbol{\epsilon}}) \quad\text{where } \overline{\boldsymbol{\epsilon}} = \overline{\mathbf{B}}\boldsymbol{U}. \tag{19}$$

Due to the vanishing of the second term, the functional of Eq. (1) can be simplified with the introduction of the modified strain energy as

$$\overline{\Pi}(\boldsymbol{u}) = \overline{\Psi}(\boldsymbol{u}) - \mathcal{W}. \tag{20}$$

We have thus obtained a single-field functional.