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Finite element formulation of various four unknown shear deformation theories for functionally graded plates

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ABSTRACT

In this paper, finite element formulation of various four unknown shear deformation theories is presented for the bending and vibration analyses of functionally graded plates. The present theories have strong similarity with the classical plate theory and accounts for shear deformation effects without using any shear correction factors. A four-node quadrilateral finite element is developed using Lagrangian and Hermitian interpolation functions to describe the primary variables corresponding to the in-plane displacements and transverse displacement, respectively. Material properties are assumed to be graded in the thickness direction according to a power-law distribution in terms of volume fractions of the constituents. Convergence test and comparison studies are performed for thin and very thick plates to demonstrate the accuracy of the present formulation.

1. Introduction

Functionally graded materials (FGMs) are heterogeneous composite materials usually made from a mixture of ceramics and metals. The mechanical properties of FGM vary smoothly and continuously from one surface to the other and thus eliminate the stress concentration at the interface of the layers found in laminated composites. One of the reasons for increasing the use of FGMs in many engineering structures is that their material properties can be tailored to different applications and working environments. The increase in FGM applications requires the development of accuracy models to predict their responses. An interesting literature review of the modeling and analysis of FGMs can be found in the paper of Birman and Byrd [1]. An updated review of more recent works on 3D exact and analytical solutions of functionally graded (FG) plates can be found in Jha et al. [2]. Thus the following literature review addresses mainly the development of finite element models based on plate theories.

Since the shear deformation effects are more pronounced in thick plates or plates made of advanced composites like FGMs, shear deformation theories such as first-order shear deformation theory (FSDT) and higher-order shear deformation theories (HSDTs) should be used to analyze FG plates. A significant number of finite element analyses of FG plates have been performed using shear deformation theories. Reddy [3] presented the finite element formulations for FG plates based on his third-order shear deformation theory (TSDT) [4]. Qian et al. [5,6] and Gilhooley et al. [7,8] employed the meshless local Petrov–Galerkin method (MLPG) and higher-order shear and normal deformable plate theory to analyze the thick FG plates. Ferreira et al. [9,10] used the TSDT and meshless collocation method with multiquadric radial basis functions to study the bending and vibration responses of FG plates. A C^0 continuous finite element based on the HSDT was developed by Pradyumna and Bandyopadhyay [11] to study the free vibration of FG curved panels. The bending and free vibration analyses of FG plates were investigated by Zhao et al. [12,13] using the FSDT and element free *kp*-Ritz method. A nine-node isoparametric finite element based on the HSDT was developed by Talha and Singh [14] to study the free vibration and bending responses of FG plates. The nonlinear bending behavior of FG plate with different boundary conditions was studied by Singha et al. [15] using the FSDT and a four-node finite element based on the exact neutral surface position. Based on the FSDT, Nguyen-Xuan et al. [16,17] investigated the bending, buckling and free vibration responses of FG plates using the edge-based smoothed finite element method (ES-FEM) [16] and node-based smoothed finite element method (NS-FEM) [17]. The bending and free vibration behavior of sandwich FG plates are studied by Natarajan and Manickam [18] using a QUAD-8 shear flexible element along with a HSDT. More recently, Valizadeh et al. [19] and Tran et al. [20] used the non-uniform rational B-spline (NURBS) based isogeometric method to study the static and dynamic responses of FG plates. Mantari and Guedes Soares [21] presented a HSDT and its finite element formulation for bending analysis of FG plates.

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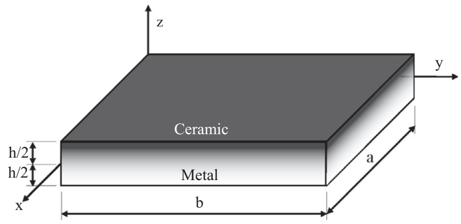

Fig. 1. Geometry and coordinate of a FG plate.

It should be noted that the FSDT gives acceptable results, but it requires a shear correction factor. Although the HSDTs do not require a shear correction factor, their equations of motion are more complicated than those of the FSDT. Therefore, Shimpi [22] develops a simple plate theory by separating the transverse displacement into bending and shear parts. The most interesting feature of Shimpi's theory is that it has fewer unknowns and governing equations than the FSDT and does not require a shear correction factor. Based on Shimpi's work, several four unknown shear deformation theories were developed using different shape functions such as polynomial functions [23–30], sinusoidal functions [31] and hyperbolic functions [32]. However, these works [23–34] are limited to the analytical solutions of simply supported plates or plates with two opposite edge simply supported and remaining ones arbitrary. The development of finite element model of these four unknown shear deformation theories is therefore necessary.

The aim of the present work is to develop a displacement-based finite element model of various four unknown shear deformation theories for the bending and free vibration analyses of FG plates with arbitrary boundary conditions. A C^1 continuous four-node quadrilateral plate element with ten degrees of freedom per node is employed. Lagrangian linear interpolation functions are used to describe the primary variables corresponding to the in-plane displacements whereas Hermitian cubic interpolation functions are considered for the transverse displacement. Convergence test and comparison studies are performed for thin and very thick plates to demonstrate the accuracy of the present formulation. Numerical results are presented to examine the effects of volume fraction index, thickness ratio and boundary condition on the responses of FG plates.

2. Four unknown shear deformation theories

Consider a rectangular FG plate with the length a, width b, and thickness h. The x-, y-, and z-coordinates are taken along the length, width, and height of the plate, respectively, as shown in Fig. 1. The displacement field of various four unknown shear deformation theories is derived based on the following assumptions: (1) the in-plane and transverse displacements are partitioned into bending and shear components; (2) the bending components of in-plane displacements are similar to those given by classical plate theory (CPT); and (3) the shear components of in-plane displacements give rise to higher-order variations of shear strains and hence to shear stresses through the plate thickness in such a way that the shear stresses vanish on the top and bottom surfaces. Based on these assumptions, the displacement field of various four unknown shear deformation theories is given in a general form as

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\ u_2(x,y,z,t) &= v(x,y,t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\ u_3(x,y,z,t) &= w_b(x,y,t) + w_s(x,y,t) \end{aligned} \tag{1}$$

where u and v are the in-plane displacements on the middle plane along the coordinates (x, y); w_b and w_s are the bending and shear components of transverse displacement, respectively; and $f(z)$ is a shape function determining the distribution of the transvese shear strains and shear stresses through the plate thickness. The shape functions $f(z)$ are chosen to satisfy the stress-free boundary conditions on the top and bottom surfaces of the plate, thus a shear correction factor is not required. In this study, three different shape functions derived by different researchers are considered: polynomial functions based on Shimpi [22]

$$f(z) = -\frac{z}{4} + \frac{5}{3}\frac{z^3}{h^2} \tag{2a}$$

sinusoidal functions based on Touratier [35]

$$f(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \tag{2b}$$

and hyperbolic sine functions based on Soldatos [36]

$$f(z) = z - h\sinh\frac{z}{h} + z\cosh\frac{1}{2} \tag{2c}$$

In addition, the four unknown FSDT (i.e.,$f(z) = 0$) recently developed by Thai and Choi [33,34] is also considered in this study. It is worth noting that the works of Thai and Choi [33,34] are only limited to analytical solutions plates with simply supported boundary conditions, whereas the present work deals with finite element solutions of plates with arbitrary boundary conditions.

The non-zero strains are given as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix}$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{3}$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \tag{4a}$$

$$g(z) = \begin{cases} 1 - f'(z) & \text{for } f(z) \neq 0 \\ \kappa & \text{for } f(z) = 0 \end{cases} \tag{4b}$$

with κ^2 being a shear correction factor.

Equations of motion of various four unknown shear deformation theories derived from Hamilton's principle are

$$\delta u : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\ddot{u} - I_1\frac{\partial \ddot{w}_b}{\partial x} - J_1\frac{\partial \ddot{w}_s}{\partial x} \tag{5a}$$

$$\delta v : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0\ddot{v} - I_1\frac{\partial \ddot{w}_b}{\partial y} - J_1\frac{\partial \ddot{w}_s}{\partial y} \tag{5b}$$

$$\begin{aligned} \delta w_b : & \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + q \\ & = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) - I_2\nabla^2\ddot{w}_b - J_2\nabla^2\ddot{w}_s \end{aligned} \tag{5c}$$