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Deslauriers–Dubuc interpolating wavelet beam finite element

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ABSTRACT

This paper presents the formulation of beam finite elements based on Deslauriers–Dubuc interpolating wavelets, also known as Interpolets. Unlike other wavelet families like Daubechies, Interpolets possess rational filter coefficients, are smooth, symmetric and therefore more suitable for use in numerical methods. Displacement and rotation shape functions are obtained and presented graphically. Expressions for stiffness matrix and force vector are developed based on connection coefficients, which are inner products of basis functions and their derivatives. In order to validate the formulation, several examples with increasing level of complexity are tested and results are compared with analytical and standard beam element solutions.

1. Introduction

The Finite Element Method (FEM) is a versatile tool for solving numerical problems. Its main advantage over other methods is its geometrical flexibility, which allows the use of complex and irregular meshes. Another known advantage of the FEM is how it naturally deals with boundary conditions.

Compactly supported wavelets emerged from the need to find new ways of representing functions with a finite number of components, especially the ones with high gradients and singularities, other than the traditional Fourier analysis, which requires an infinite number of components for exact representation. Later, it was shown that wavelets have several properties that are especially useful for representing solutions of partial differential equations (PDEs), such as orthogonality, compact support and exact representation of polynomials of a certain degree [1]. These characteristics allow the efficient and stable calculation of functions at different levels of resolution, both in time and frequency domain.

In the last three decades, wavelet functions have been widely used in the solution of numerical problems, such as image compression and financial analysis. More recently, these functions have been applied in numerical schemes such as the FEM. Among these applications are thermal conduction analyses [2] and wave propagation problems [3,4].

The conventional formulation of the FEM uses polynomials for interpolating the displacement within the elements (shape functions). This work proposes the use of a different family of functions called Interpolating Wavelets as basis functions in order to obtain satisfactory results in terms of stability and convergence with less refined meshes than the traditional FEM would require.

The most commonly used wavelet family of functions is the one developed by Ingrid Daubechies [5]. The mathematical foundations for the wavelet theory were formulated for Daubechies wavelets and then extended to other families.

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. This mother function is defined using a recursive linear combination of these basis functions, whose weights are called filter coefficients. This recursive scheme is called pyramidal algorithm [6]. Although many applications use only the wavelet filter coefficients, there are some which explicitly require the values of the basis functions and their derivatives at some target points.

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable [7]. Those integrals are called connection coefficients and they appear naturally in a Finite Element scheme. Due to some properties of wavelet functions, these coefficients can be

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obtained by solving an eigenvalue problem using filter coefficients.

Working with dyadically refined grids, Deslauriers and Dubuc [8] obtained a new family of wavelets with interpolating properties, later called Interpolets. Their filter coefficients are obtained from the autocorrelation of the same order Daubechies' coefficients. As a consequence, interpolets are symmetric and smooth, which is especially interesting in numerical analysis. Another advantage of Interpolets is that their representation of a function as a weighted sum leads to coefficients which are evaluations of the same function at target points on a dyadic mesh. Additionally, the number of vanishing moments of Interpolets is greater than Daubechies' of the same order, which allows exact representation of higher degree polynomials. The adoption of interpolating wavelets in numerical methods has been presented in some works which have combined wavelets and modified scaling basis functions in a FE scheme [9,10].

The formulation of an Interpolet-based Finite Element system is demonstrated for one-dimensional beam problems and a general approach allows different orders of scaling functions to be tested. Shape functions, stiffness matrices and equivalent nodal forces are explicitly derived using scaling functions without additional modifications. Several examples with increasing level of complexity were studied successfully. Results were accurate not only for displacements but also for rotations and bending moments if compared with standard beam finite elements.

2. Wavelet theory

Unlike trigonometric functions, the distinguishing feature about wavelets is that they are localized in space, which allows local variations of the problem to be analyzed at various levels of resolution. Thus, multiresolution analysis using orthogonal, compactly supported wavelets has been successfully applied in numerical simulation.

All the mathematical foundation for wavelet functions is based on the algorithms for Daubechies wavelets. Wavelet basis are composed of two kinds of functions: scaling functions (φ) and wavelet functions (ψ). The two combined form a complete Hilbert space of square integrable functions. The spaces generated by scaling and wavelet functions are complementary and both are based on the same mother function. In the following expressions, known as the two-scale relation, a_k are the scaling function filter coefficients and N is the Daubechies wavelet order [11].

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k)$$

$$\psi(x) = \sum_{k=0}^{N-1} (-1)^k a_{N-1-k} \varphi(2x-k) \tag{1}$$

In general, there are no analytical expressions for wavelet functions, which can be obtained using iterative procedures like Eq. (1). In order to comply with the requirements of orthogonality and compact support, wavelets present, in general, an irregular fractal-like shape. Fig. 1 shows Daubechies scaling function of order $N=4$.

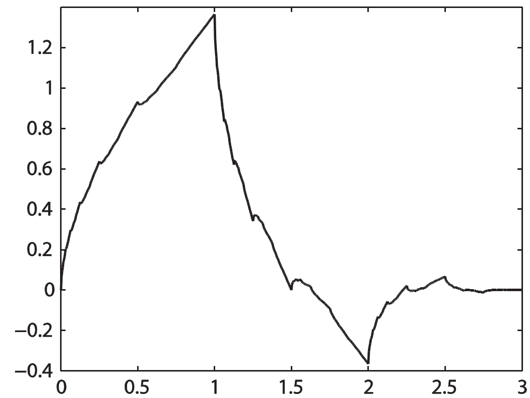

Fig. 1. Daubechies scaling function of order $N=4$.

2.1. Deslauriers–Dubuc interpolets

The term interpolet was first used by Donoho [12] to designate wavelets with interpolating characteristics. The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition [13]:

$$\varphi(k) = \delta_{0,k} = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}, \quad k \in \mathbb{Z} \tag{2}$$

Any function $f(x)$ can be represented as a linear combination of the basis functions at level of resolution j:

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(2^j x-k) = \sum_{k\in\mathbb{Z}} c_k \varphi_{j,k}(x) \tag{3}$$

Evaluating the function at a dyadic grid point $x=2^{-j}k$ yields:

$$f(2^{-j}k) = \sum_{r\in\mathbb{Z}} c_r \varphi_{j,r}(2^{-j}k) = \sum_{r\in\mathbb{Z}} c_r \varphi(2^j 2^{-j}k-r) = \sum_{r\in\mathbb{Z}} c_r \delta_{k-r} = c_k \tag{4}$$

This characteristic is very interesting for numerical applications, since a function can be represented as a weighted sum in which the coefficients are evaluations of the same function at target points on a dyadic mesh:

$$f(x) = \sum_{k\in\mathbb{Z}} f(2^{-j}k)\varphi(2^j x-k) \tag{5}$$

2.2. Interpolet properties

The set of properties summarized in Eq. (6) is valid for Deslauriers–Dubuc interpolating wavelets. Some of these properties, like compact support and unit integral are shared with Daubechies wavelets.

$$\mathrm{supp}(\varphi) = [1-N, N-1]$$
$$\int_{-\infty}^{+\infty} \varphi(x)dx = 1$$
$$\int_{-\infty}^{+\infty} x^k \psi(x)dx = 0, \quad k=0,\ 1, \ldots,\ N-1 \tag{6}$$

On the other hand, orthogonality is not fully satisfied for interpolating wavelets (Eq. 7), although this is not a crucial issue.

$$\int_{-\infty}^{+\infty} \varphi(x-i)\varphi(x-j)dx = \int_{-\infty}^{+\infty} \varphi_i(x)\varphi_j(x)dx \neq \delta_{i,j} \tag{7}$$

The last expression in Eq. (6) derives from the vanishing moments property, which states that polynomials $q(x)$ of degree up to $N-1$ can be exactly represented in a closed interval as a linear combination of Deslauriers–Dubuc scaling functions of order N:

$$q(x) = a_1 + a_2 x + \cdots + a_{m+1}x^m = \sum_k q(2^{-j}k)\varphi_{j,k}(x), \quad m \leq N-1 \tag{8}$$

In Eq. (8), index k varies according to the translations needed to cover the interval within which the representation is intended. The number of degrees of freedom (DOFs) of a wavelet scaling function is given by the number of translations needed to cover a